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A STUDY OF THE OPTIMUM VELOCITY CHANGE TO INTERCEPT AND RENDEZVOUS

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SUMMARY

From an arbitrary set of initial conditions, the problem of determining the minimum velocity change to (a) intercept and (b) rendezvous with an orbiting target is investigated by using an orbital mechanics approach. Certain approximate solutions are found in terms of the optimum time to intercept and rendezvous, and the physical significance of these solutions is determined in the region where they are valid. However, for practical problems of midcourse guidance where the solutions must be valid over long periods of time (or more correctly, over large values of the anomaly) a need for more exact solutions is indicated.

INTRODUCTION

In the past several years there has been much interest shown in the optimization of the velocity changes required to make an orbit-to-orbit transfer. A more difficult problem is that of making an orbital transfer maneuver such that the transfer vehicle or ferry arrives at the new orbit at the same time as a second vehicle already established in that orbit. This latter problem is directly applicable to rendezvous and exists whenever a transfer vehicle is launched either directly from the earth or from an existing orbit. In general, the problem is that of finding that minimum velocity change required to go from an arbitrary set of initial conditions to a specified time-variant set of end conditions which represent rendezvous with either a real or an imaginary target.

This general problem, and in particular its application to rendezvous, is the subject of this paper. In the general case, the ferry vehicle is in the vicinity (several hundred miles) of an orbiting target and it is desired to intercept and rendezvous with that target. If the ferry vehicle is not on an interception course, a velocity correction must be applied to establish such a course. Since there are an infinite number of such courses, each associated with a time until interception, the one requiring the least fuel may be the most desirable, and the time associated with it is referred to hereafter as the optimum time to intercept. An expression for this optimum velocity correction was given in reference 1.

When interception does occur, a final velocity change must be made to bring the relative velocity between target and ferry to zero. This final velocity change effectively injects the ferry into the same orbit as the target. When the sum of the velocity changes required for both the intercept and terminal maneuvers is minimized, the least fuel for the complete rendezvous maneuver is obtained. The intercept trajectory obtained by this latter technique is usually not the same as the optimum trajectory to intercept. The time associated with the least fuel consumption to rendezvous is hereafter referred to as the optimum time to rendezvous. Some recent work on this subject was done by H. Hornby of the NASA Ames Research Center[1], and in the appendix of this paper some approximate solutions obtained by Hornby and the author are derived.

Although the general problem is essentially nonlinear, approximate linear solutions can be obtained and analyzed for general characteristics. These general characteristics are formulated herein. In presenting the material, (1) the development of some approximate expressions for the optimum time to intercept and the optimum time to rendezvous is briefly outlined, (2) the validity and significance of these approximate expressions is discussed, and (3) some results obtained from using these expressions in a rendezvous guidance system are described.

SYMBOLS

The English system of units is used in this study. In case conversion to metric units is desired, the following relationships apply: 1 foot = 0.3048 meter, and 1 statute mile = 5,280 feet = 1,609.344 meters.

a	acceleration
G	universal gravitation potential
g	acceleration due to gravity, 32.2 ft/sec
$\bar{i},\bar{j},\bar{k}$	unit vectors in x,y,z directions, respectively
K	square of tangential component of relative velocity V
M	mass of earth

[1]"Least Fuel, Least Energy and Salvo Rendezvous." A paper presented at 15th Spring Technical Conference, ARS and IRE, on April 12, 1961 at Cincinnati, Ohio.

m	mass of ferry
R,θ,ψ	orthogonal components of spherical coordinate system centered in target and directly related to x,y,z components
r	radial distance measured from center of earth
T	thrust
t	time
V	relative velocity between target and rendezvous vehicle
V_1,V_2,V_3	relative velocity immediately following ΔV_1, immediately prior to ΔV_2, and immediately following ΔV_2, respectively
ΔV	impulsive change in V
$\Delta V_1,\Delta V_2$	first and second impulsive velocity correction, respectively
ΔV_∞	impulsive change in V for an infinite time to intercept (or rendezvous)
W	weight of ferry vehicle
X,Y,Z	inertially fixed axes
x,y,z	orthogonal components of rectangular coordinate system centered in target; x and y axes lie in the plane of orbit; when rotating axes are used, rotation occurs about z (out-of-plane) axis such that y is always alined with local vertical

$\delta = 3\omega\tau \sin \omega\tau - 8(1 - \cos \omega\tau)$

ϵ	miss distance
Θ	angular distance (anomaly) around center of earth measured in orbital plane of target
μ/r	gravitational potential
τ	time until interception or rendezvous
τ_1,τ_2	optimum time until interception and rendezvous, respectively
Ω	angular velocity vector of rotating rectangular coordinate system

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ω — angular velocity of target about earth $\left(=\dot{\Theta}_s\right)$ and hence, by choice, also angular velocity of rotating coordinate system about z-axis

$\omega\tau$ — angular distance (anomaly) about earth to be traveled by target until interception or rendezvous

∇ — vector differential operator

Subscripts:

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f — ferry vehicle

min — minimum

o — initial conditions

s — target satellite

x,y,z — component with respect to x-, and y-, and z-axis, respectively

Dots over variables denote differentiation with respect to time. Bars over quantities denote vectors, and $| \; |$ denotes absolute value or magnitude of a vector quantity. Primes on variables denote desired values.

GENERAL DISCUSSION

Techniques

Rendezvous guidance or steering techniques can be catalogued[2] into two general categories: those techniques based on proportional or constant-bearing navigation and those based on orbital mechanics. These two techniques are illustrated in figure 1. In the proportional navigation technique, the angular rate of rotation of the velocity vector with respect to inertial space is controlled in proportion to the angular rate of the line of sight and gives an exponential approach to a constant-bearing course. This technique is generally used in the terminal phase of rendezvous where the thrust is continuously programed to bring the range plus some miss distance $R + \epsilon$ and range rate $\dot{R}$ to zero

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[2]"Problems and Potentialities of Space Rendezvous." Paper presented by John C. Houbolt at the International Symposium on Space Flight and Re-Entry Trajectories, organized by the IAF, on June 19-21, 1961 at Louveciennes, France.

simultaneously. A typical expression for controlling the thrust along the line of sight is

$$a = \frac{T}{W} = \frac{\dot{R}^2}{2g(R + \epsilon)}$$

In the orbital mechanics technique, illustrated in figure 1, solutions to various approximate equations of motion between the ferry and its target are used to compute a trajectory which will lead to an interception between the two bodies at some future time. This technique is more applicable for prediction where large changes in anomaly occur during the interception phase. Because of the gravitational effect of the earth, two vehicles on a collision course do not maintain a constant line of sight with respect to inertial space except during the terminal or final phase before contact (thus, the constant-bearing technique is usually applicable only during these last few minutes).

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One means of accounting for the rotational effect of the line of sight over a wide range of possible interception trajectories is to use for onboard guidance as general a set of equations of motion as is possible. Computer weight, speed, and power requirements will produce some compromise in the choice of the guidance equations. Various forms of the equations of motion of one body relative to a second body in orbit are given in table I for both rotating and inertially fixed sets of axes. Certain of the sets of differential equations have known closed-form solutions and therefore are particularly useful for guidance equations. The most general set of equations having such solutions is that attributed to Clohessy and Wiltshire (ref. 2 and others) marked as T-1 on the table:

$$\left.\begin{aligned} \ddot{x} - 2\omega\dot{y} &= \frac{T_x}{m} \\ \ddot{y} + 2\omega\dot{x} - 3\omega^2 y &= \frac{T_y}{m} \\ \ddot{z} + \omega^2 z &= \frac{T_z}{m} \end{aligned}\right\} \tag{1}$$

A set of less exact equations has been used by Hornby and is marked T-2:

$$\left.\begin{aligned} \ddot{x} - 2\omega\dot{y} &= \frac{T_x}{m} \\ \ddot{y} + 2\omega\dot{x} &= \frac{T_y}{m} \\ \ddot{z} + \omega^2 z &= \frac{T_z}{m} \end{aligned}\right\} \tag{2}$$

Both of these sets of equations yield closed-form solutions of the trajectory of the ferry relative to the target, and with the addition of some specified value of time the solutions to these equations can be used to calculate velocity corrections which will lead to interception and rendezvous.

In this paper the use of the orbital mechanics techniques for computing intercept or rendezvous trajectories requiring the minimum fuel consumption and, hence, the optimum time are investigated. Equations (1) and (2) are the two sets of equations used to compute this optimum time associated with the minimum fuel consumption. When actual trajectories are calculated, however, the more exact but nonlinear differential equations marked T-3 are used:

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$$\left.\begin{aligned}
&\ddot{x} - 2\omega\dot{y} - x\left(\omega^2 - \frac{GM}{{r_f}^3}\right) = \frac{T_x}{m} \\
&\ddot{y} + 2\omega\dot{x} - \left(y + r_s\right)\left(\omega^2 - \frac{GM}{{r_f}^3}\right) = \frac{T_y}{m} \\
&\ddot{z} + \frac{GM}{{r_f}^3}\, z = \frac{T_z}{m}
\end{aligned}\right\} \qquad (3)$$

Optimum Time to Intercept

In order to review briefly the results of reference 1, consider a rendezvous vehicle moving in the general direction of an orbiting target. With respect to the target, the position and velocity of the vehicle are known. This situation and the coordinates used to describe the motion are shown in figure 2(a), and a vector diagram of the velocities is given in figure 2(b).

If the vehicle is on a collision course with the target, no immediate action need be taken. (A collision course is established if the rendezvous vehicle is on a trajectory that will eventually lead to a collision with the target.) If, however, the vehicle is not on a collision course, it is desired to know what is the minimum velocity change required to put it on a collision course. The situation can be evaluated by using linear equations such as equations (1) to compute the instantaneous velocity components (based on position data and a choice of time to intercept) required for a collision course. In terms of the rotating rectangular coordinate system of references 1 and 2, the relative velocity components of the ferry vehicle which would be required to intercept at future time τ are

$$
\left.
\begin{aligned}
\frac{\dot{x}_o'}{\omega}(\tau) &= \frac{1}{\delta}\Big[x_o \sin \omega\tau + y_o(6\omega\tau \sin \omega\tau - 14 + 14 \cos \omega\tau)\Big] \\
\frac{\dot{y}_o'}{\omega}(\tau) &= \frac{1}{\delta}\Big[2x_o(1 - \cos \omega\tau) + y_o(4 \sin \omega\tau - 3\omega\tau \cos \omega\tau)\Big] \\
\frac{\dot{z}_o'}{\omega}(\tau) &= -z_o \cot \omega\tau
\end{aligned}
\right\} \qquad (4)
$$

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where $\delta = 3\omega\tau \sin \omega\tau - 8(1 - \cos \omega\tau)$ and where x_o, y_o, and z_o give the instantaneous position of the vehicle relative to the target and ω is the angular velocity of the target about the earth. Primes have been used to denote that these are the velocity components required for interception and not necessarily equal to the actual velocity components $\dot{x}_o$, $\dot{y}_o$, and $\dot{z}_o$. The required velocity change $\Delta\bar{V}_1$ is the difference between the present velocity V_o and the required velocity V_1 and depends upon a choice of a value for the time to intercept τ:

$$\Delta\bar{V}_1(\tau) = \bar{V}_1(\tau) - \bar{V}_o \qquad (5)$$

or

$$\left|\Delta V_1(\tau)\right| = \sqrt{\left[\dot{x}_o'(\tau) - \dot{x}_o\right]^2 + \left[\dot{y}_o'(\tau) - \dot{y}_o\right]^2 + \left[\dot{z}_o'(\tau) - \dot{z}_o\right]^2} \qquad (6)$$

If the required velocity change ΔV_1 is computed for all possible values of τ, the result is similar to that shown in figure 3. The minimum velocity change occurs when

$$\frac{d\left|\Delta V_1(\tau)\right|}{d\tau} = 0 \qquad (7)$$

and the time to intercept for this condition, referred to as the optimum time to intercept, is given by the approximate expression:

$$\tau_1 = \frac{x_o^2 + y_o^2 + z_o^2}{-\left(x_o\dot{x}_o + y_o\dot{y}_o + z_o\dot{z}_o\right)} = \frac{R_o}{-\dot{R}_o} \qquad (8)$$

where R_o is the instantaneous line-of-sight distance between the vehicle and the target, and $\dot{R}_o$ is the time rate of change of that distance. It was shown in reference 1 that this expression is valid for values of $\omega\tau$ (the anomaly to interception) less than 40^o.

If the vehicle is on a collision course, the minimum of the curve in figure 3 would be tangent to the line $\Delta V_1 = 0$ at time $\tau = \tau_1 = R_o/-\dot{R}_o$. If the vehicle is not on a collision course, the minimum velocity change to get on a collision course would be obtained by choosing a time to intercept $\tau = R_o/-\dot{R}_o$ and computing the required velocity components from equations (4).

Physically, this solution indicates that the minimum velocity change to intercept would be obtained if the angular rate of the line of sight were corrected without changing the range rate.

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Optimum Time to Rendezvous

After the vehicle has been put on a collision course (by modifying V_o by ΔV_1 at $t = 0$), it will be necessary to bring the relative velocity between vehicle and target to zero as $R \rightarrow 0$. At time τ later, interception will occur and the components of the relative velocity will be:

$$\left.\begin{aligned}
\frac{\dot{x}_1}{\omega}(\tau) &= \frac{1}{\delta}\left[x_o \sin \omega\tau + 2y_o(1 - \cos \omega\tau)\right] \\
\frac{\dot{y}_1}{\omega}(\tau) &= \frac{1}{\delta}\left[2x_o(\cos \omega\tau - 1) + y_o(4 \sin \omega\tau - 3\omega\tau)\right] \\
\frac{\dot{z}_1}{\omega}(\tau) &= -z_o \csc \omega\tau
\end{aligned}\right\} \tag{9}$$

If at interception the relative velocity is brought to zero[3], the magnitude of the velocity change is given by the expression

$$\left|\Delta V_2(\tau)\right| = \sqrt{\left[\dot{x}_1(\tau) - 0\right]^2 + \left[\dot{y}_1(\tau) - 0\right]^2 + \left[\dot{z}_1(\tau) - 0\right]^2} \tag{10}$$

Thus if it is desired to minimize the total velocity change required to intercept and rendezvous, it is necessary to calculate the minimum value of the scalar sum $|\Delta V_1| + |\Delta V_2|$. For a minimum,

$$\frac{d}{d\tau}\left(|\Delta V_1| + |\Delta V_2|\right) = 0 \tag{11}$$

[3]In theory only. In practice, the braking maneuver would start some time prior to interception.

When the approximate differential equations of Hornby are used (see appendix), condition (11) is satisfied by the following expression for optimum time to rendezvous:

$$\tau_2 = \frac{R_o}{-\dot{R}_o} \pm \sqrt{\frac{R_o^2(\dot{\theta}_o - \omega)^2\cos^2\psi_o + R_o^2\dot{\psi}_o^2}{\omega^2\dot{R}_o^2}} \tag{12}$$

Since $\dot{R}$, $R(\theta - \omega)\cos\psi$, and $R\dot{\psi}$ are the components of the relative velocity in spherical coordinates, it may be seen from equation (12) the the value of τ_2 is dependent only on the relative velocity and range and not on the orientation of the gravity field. The fact that τ_2 is not dependent on orientation is probably due to the degree of approximation.

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In special cases where ferry vehicle and target are in the same orbital plane, then $\psi = \dot{\psi} = 0$ and

$$\tau_2 = \frac{R_o}{-\dot{R}_o} \pm \left|\frac{R_o(\dot{\theta}_o - \omega)}{\omega\dot{R}_o}\right| \tag{13}$$

It can be seen in both equations (12) and (13) that the first term on the right is simply τ_1 and the second term is

$$\frac{\left|\overline{V}_{tangential}\right|}{\omega\left|\overline{V}_{radial}\right|}$$

with respect to an inertially fixed set of axes in either of the two vehicles. The term $\left|\dot{\theta}_o - \omega\right|$ is simply the difference between the measured angular velocity of the line of sight in the plane of rotation $\dot{\theta}_o$ and the known angular velocity of the line of sight due to the rotation of the coordinate system ω. The $\left|\dot{\theta}_o - \omega\right|$ term and $\dot{\psi}$ are zero when the ferry is moving directly toward the target. When $\dot{\theta} \neq \omega$ or $\dot{\psi} \neq 0$, the line of sight between the ferry and target will be rotating with respect to inertial space.

It can thus be immediately surmised that equations (12) and (13) for τ_2 have the same limitations as equation (8) for τ_1, and that they are valid only during the terminal phase where a constant-bearing course is near optimum. It is shown subsequently that numerically this limitation means an anomaly until rendezvous of 40^o or less.

Although limited in scope, a study of these equations within the range of validity should give some insight into the proper or optimum course of action when the ferry is not on a collision or intercept course. Such a study may also suggest expressions valid over a wider range of conditions. In the remainder of this paper, these possibilities are explored.

NUMERICAL EXAMPLES AND RESULTS

Range of Validity

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As a means of evaluating the expressions derived in the previous section, the following techniques were employed. With the nonlinear differential equations (eqs. (3)), exact rendezvous trajectories were calculated in reverse time by using a digital computer. That is, the ferry was placed at the target ($x = y = z = 0$) and arbitrarily assigned a relative velocity of 1,000 ft/sec. The equations of motion were then solved in negative time until the ferry and target were separated by several hundred miles. Both coplanar and noncoplanar collision course trajectories were obtained in this manner. Relative positions and velocities at several positions along the trajectories were then picked to evaluate the accuracy of the various approximate solutions. The conditions for three such positions along the trajectories are given in table II. In all cases the target was in a circular orbit 300 statute miles above a spherical earth (with radius of 3,960 statute miles) and had an angular velocity ω of 0.0011122 radian /sec.

With the specified initial conditions given in table II, the variations in $|\Delta V_1|$, $|\Delta V_2|$, and $|\Delta V_1| + |\Delta V_2|$ were calculated as a function of $\omega\tau$ for each of the three cases by using equations (4), (6), (9), and (10) (referred to as first-order gravity equations) and by using equations (A9) and (A10) of the appendix (referred to as uniform gravity equations). Also, the values of $\omega\tau_1$ and $\omega\tau_2$ were calculated by use of equations (8) and (12). The results are shown in figure 4.

Case I is shown in figure 4(a). For the given initial conditions, collision will occur at a value of $\omega\tau$ of 15.8°, and this condition is indicated on the absissa of the figure by an arrow pointing upward. The calculated values of $\omega\tau_1$ and $\omega\tau_2$ are indicated by arrows pointing downward. It may be seen that both sets of approximate equations give reasonably good results. The minimum for $|\Delta V_1|$ obtained by using the uniform gravity equations occurs at

$$\omega\tau_1 = \tan^{-1}\left(\frac{\omega R_o}{-\dot{R}_o}\right)$$

and the minimum for $|\Delta V_1|$ obtained by using the first-order gravity equations occurs at

$$\omega\tau_1 = \frac{\omega R_o}{-\dot{R}_o}$$

To be exactly correct, the $|\Delta V_1|$ curve should be tangent to $\Delta V = 0$ and the $|\Delta V_1| + |\Delta V_2|$ curve should be tangent to $\Delta V = 1{,}000$ ft/sec at the value of $\omega\tau$ indicated by the arrow pointing upward. As would be expected by the degree of approximation, the first-order gravity equations give the more accurate results.

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Shown on the extreme right of the figure is a circle marked $|\Delta V_\infty|$. This value of ΔV represents the magnitude of the relative initial velocity and was obtained from the following calculation:

$$|\Delta V_\infty| = \sqrt{\dot{R}_o^{\,2} + R_o^{\,2}(\dot{\theta}_o - \omega)^2\cos^2\psi_o + (R_o\dot{\psi}_o)^2}$$

It has been suggested in reference 5 that, in the absence of gravity, the optimum fuel consumption would be obtained by removing all of the relative velocity except for an infinitesimal amount directed toward the target. If rigorously applied, this maneuver would require an infinite time to intercept (or rendezvous) under a zero-gravity condition. However, as a measure of the minimum ΔV, this simple calculation appears to be reasonably good in the near vicinity of the target.

Case II is shown in figure 4(b). The ferry is initially 23.4 statute miles out of the target's orbital plane, and on its present course will intercept at a value of $\omega\tau = 32.1^\circ$. The differences in the two sets of approximate equations are more pronounced than for Case I because of the longer time to intercept. The computed value of $\omega\tau_2$ is greater than the computed value of $\omega\tau_1$ because of the out-of-plane motion.

Case III is shown in figure 4(c). The ferry is in the target's orbital plane and will intercept at a value of $\omega\tau = 64.7^\circ$. At this large anomaly, the uniform gravity approximation is inadequate, as are the expressions for $\omega\tau_1$ and $\omega\tau_2$. The first-order gravity equations still produce a minimum $|\Delta V|$ near the correct value, but are also beginning to deteriorate.

Based on the physical situation, it would be expected that when the vehicle is on a collision course the minimum of the exact $|\Delta V_1|$ and $|\Delta V_2| + |\Delta V_2|$ curves would occur at exactly the same anomaly.

This statement indicates that the optimum course of action for either interception or rendezvous is to make no change in velocity until ΔV_2 is applied. In general, the results of figure 4 support this expectation. At small values of the anomaly where the approximate equations are most accurate, both the uniform and first-order gravity equations show good agreement in this respect. At the larger values of the anomaly, the deviation from the exact condition is due to the degree of approximation. The data of figure 4 also show that the approximate expressions for these ΔV minimums, $\omega\tau_1$ and $\omega\tau_2$, are only good at small angles where relationships such as $\sin \omega\tau \approx \omega\tau$ are applicable. Since these expressions are also exact solutions to the uniform gravity equations, they fall within this degree of approximation.

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It is important to note that the $|\Delta V_1| + |\Delta V_2|$ curve has a very steep slope prior to its minimum and a very shallow slope thereafter. Thus, overestimating the optimum time to rendezvous is not as expensive as underestimating it. Unfortunately, the approximate values of $\omega\tau_1$ and $\omega\tau_2$ generally underestimate the anomaly associated with the true minimums of the ΔV curves and thus lead to large, unnecessary velocity changes and shorter times to intercept.

Offcourse Corrections

In order to determine what types of corrections are called for when the ferry is not on a collision course and what sign (±) should be used in equations (12) and (13), cases were calculated over a range of offcourse conditions. Typical results are illustrated in figure 5.

As a basis for comparing the results for the offcourse trajectories, figure 5(a) is simply a repeat of the results for the oncourse condition (Case I of table II) which are plotted as figure 4(a). In figures 5(b) and 5(c), the initial position and the initial value of $\dot{R}_o$ were held constant and only the initial value of $R_o\dot{\theta}_o$ was changed to give a nonintercept trajectory. In figure 5(b), the ferry has been put off course by increasing $R_o\dot{\theta}_o$ by 411.2 ft/sec. This error in velocity, if not corrected, would cause the ferry to pass 10 to 15 statute miles in front of the target. In figure 5(c), $R_o\dot{\theta}_o$ has been decreased by the same amount. This error would lead to an equal miss distance behind the target if not corrected. In each part of figure 5 the situation has been sketched at the top of the figure, $\dot{R}_o$ being the radial (line-of-sight) or closing velocity component and $R_o(\dot{\theta}_o - \omega)$ being the tangential velocity component taken with respect to inertially fixed (nonrotating) axes in the target.

Also shown on each part of figure 5 are: (1) the calculated values of $\omega\tau_1$ and $\omega\tau_2$, (2) the computed velocity changes $\left(\overline{\Delta\dot{R}_1},\ \overline{\Delta R\dot{\theta}_1}, \text{ and } \overline{\Delta\dot{R}_2}\right)$ obtained from the first-order gravity equations at the calculated values of $\omega\tau_1$ and $\omega\tau_2$, and (3) the velocity change indicated by the zero-gravity approximation, $|\Delta V_\infty|$.

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Figure 5 illustrates several characteristics of the ΔV variations. First, the minimum of the $|\Delta V_1| + |\Delta V_2|$ curve always occurs at a larger value of $\omega\tau$ than does the minimum of the $|\Delta V_1|$ curve. Hence, it would appear that the two terms in equations (12) and (13) should be added. Second, if one allows for the fact that the computed value of $\omega\tau_1$ is not the exact minimum but usually underestimates the time (or anomaly) of the exact minimum, then it would appear from observation of figure 5 that the true minimum of the $|\Delta V_1|$ curve occurs when $\Delta\dot{R} = 0$ or $\dot{R}_o{}'(\omega\tau) = \dot{R}$. The data suggest that for interception prior to this minimum, $\overline{\Delta\dot{R}}$ would be negative and an increase in the line-of-sight velocity would be required; for interception at a later time, $\overline{\Delta\dot{R}}$ would be positive and a decrease in $\dot{R}$ would be required. For most practical cases this may be a sufficiently accurate statement, but detailed study indicates that it is not exactly true. The reason for this observation is illustrated in figure 6 with the use of a hodograph presentation. In order to show more clearly the source of this data, the presentation is first given in terms of the rotating x,y,z coordinates and then in terms of the nonrotating R,θ,ψ coordinates.

In figure 6(a), a V_1 and a V_2 curve have been plotted as a function of the components $\dot{x}$ and $\dot{y}$. The construction of these curves depends only on the initial positions $(x_o,\ y_o, \text{ and } z_o)$ given by Case I of table II and the variable $\omega\tau$. Every point on the curves represents a specified value of $\omega\tau$, and is simply a plot of equations (4) and (9) ($\dot{z}$ and z_o being zero for all values of $\omega\tau$). Also plotted on figure 6(a) with circled points are the initial velocities $\dot{x}_o$ and $\dot{y}_o$ considered in the discussion of figure 5. The point lying on the V_1 curve represents the initial conditions of figure 5(a), the point above the line represents the initial conditions of figure 5(b), and the point below the line applies to figure 5(c). Thus any point on this two-dimensional space represents a possible set of initial velocities, but only those velocities which fall on the V_1 curve will lead to an interception. The anomaly $\omega\tau$ or time τ until interception will vary, but is uniquely defined at each point on the V_1 curve. Thus for any set of initial velocities which do not fall on the V_1 curve, the minimum velocity change to intercept is defined as the shortest vector that can be drawn from that point to the V_1 curve. This vector is the minimum $\Delta\overline{V}_1$, and the τ that corresponds to this minimum velocity change is (by definition) the optimum time to intercept.

Since every point on the V_1 curve represents an interception at some later time τ, then there corresponds a relative velocity at interception given by the V_2 curve. For illustration purposes several of these corresponding points have been joined by straight dashed lines (even though obviously, the variation in $\dot{x}$ and $\dot{y}$ between corresponding points are not linear) and the anomaly until rendezvous noted for each set of points so marked. The $\Delta\overline{V}_2$ velocity change is then the vector joining any point on the V_2 curve and the origin. The scalar sum of the ΔV_1 and ΔV_2 vectors represents the total velocity change to rendezvous, and the value of τ corresponding to the minimum of this scalar sum is (by definition) the optimum time to rendezvous.

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Figure 6(b) illustrates how these V_1 and V_2 curves vary during a rendezvous approach trajectory. The curves were constructed from the three positions given by Cases I, II, and III of table II (using only the in-plane data from Case II). The relative initial velocities for each case are noted on the figure with circled points, and the anomaly until intercept noted. The relative velocities at intercept are those indicated by the circled point marked $\omega\tau = 0^{\circ}$. A curve passing through these points would give the variation of the relative velocities during the collision-course trajectory.

These same data are presented in figures 6(c) and 6(d) in terms of the polar coordinates $-\dot{R}$ and $R(\dot{\theta} - \omega)$. In this coordinate system, $\overline{V}_2$ always lies along the $-\dot{R}$ axis since at collision there cannot be any relative angular velocity between the two bodies in the inertial coordinate system.

Figure 6(c) illustrates why, if the vehicle is offcourse, the minimum velocity change to establish a collision course does not correspond precisely to $\Delta\dot{R} = 0$. Because of the curvature of the V_1 line, the shortest vector from any point not on the line itself will have a small $\dot{R}$ component. This fact is illustrated in figure 6(c) by the vectors drawn from the two offcourse cases. For the cases considered, the $\dot{R}$ component was always small to the point where, for practical considerations, it could be ignored.

The increased curvature of the V_1 and V_2 curves as $\omega\tau$ approaches 90° may be due to the inaccuracy of the approximate equations in this region. However, the true shapes of these lines in this region have not been established, and more exact solutions may exhibit even more curvature as the nonlinearities at large times to rendezvous are taken into account.

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For the two offcourse conditions illustrated in figures 5 and 6, it was found that the minimum total velocity change to rendezvous $|\Delta V_1| + |\Delta V_2|$ led to trajectories that were tangent or very nearly tangent to the orbit of the target at the time of rendezvous. This result is shown in figure 7, which is a copy of figure 6(a) with the ΔV_1 and ΔV_2 vectors drawn in for the three conditions of figure 5. The initial or oncourse trajectory shown in figure 5(a) would have terminated at the target with a closing velocity of 1,000 ft/sec at an angle $\left(\tan^{-1} \frac{\dot{y}}{\dot{x}}\right)$ of 30°. The two offcourse trajectories, after correction ΔV_1, would have terminated at the target with velocities close to 400 ft/sec with virtually no $\dot{y}$ component of velocity. The calculated values of $\omega\tau_2$ were used to establish the optimum time in each case. Whether this characteristic is generally true has not been established. However, it would not be surprising to find that the minimum velocity for a two-impulse rendezvous should lead to a transfer trajectory tangent to or very nearly tangent to the target's orbit at the point of interception. Further study to establish the physical character of the transfer trajectory appears merited.

Rendezvous With a Guidance System and Multiple Corrections

So far, only static situations have been discussed. However, in reference 6, a guidance system is described for maintaining a ferry vehicle on a collision course by making velocity corrections every time the line-of-sight range is halved. The guidance system was simulated on a high-speed digital computer and the effect of various parameters evaluated by using τ_1 to compute the time to rendezvous. With this same setup, but with the assumption of a perfect radar (exact values of position and velocity were supplied to the guidance system) and no dead band in the guidance system, the effect of using τ_2 rather than τ_1 was evaluated for two typical cases. The initial conditions for the cases considered are given in table III. The target was again in a 300-mile circular orbit, and the ferry was launched to arrive at the target with an in-plane relative velocity of 600 ft/sec after 1,240 seconds ($\omega\tau = 81.06°$). The trajectory calculations were stopped when the range rate $\dot{R}$ became zero, at which time the line-of-sight range R was between 40 and 120 feet. In Case IV, the equation for τ_1 (eq. (8)) was used with the guidance equation whenever $\omega\tau_1 \leqq 40°$. In Case V, the equation for τ_2 (eq. (13)) was used whenever $\omega\tau_2 \leqq 40°$.

In Case VI, the ferry was initially 50 miles out of plane, and the complete equation for τ_2 (eq. (12)) was used.

The velocity corrections required to complete the rendezvous in each case are given in part (b) of table III. The first item is the vector sum of the in-plane ($\dot{x}$ and $\dot{y}$) velocity changes during the midcourse phase. The second item is the resultant velocity at the time of interception and represents the terminal velocity change required to match orbits. The sum of these two items have been placed in the third row. In Case VI, the ferry was initially out of plane, and the velocity change chargeable to putting the ferry into the same plane as the target is listed in the fourth row. The total of the velocity changes is given in the fifth row, and the time and anomaly actually required for the rendezvous are given as the last two items for each case.

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The results show that the expression for τ_2 gave better results than did the expression for τ_1. The difference between the total in-plane velocity change required with τ_1 and that with τ_2 was about 120 ft/sec. When the ferry was initially out of plane by 50 miles (Case VI), a larger percent of the total fuel used went into the midcourse correction, but the sum of the in-plane corrections was virtually the same as in Case V. This difference in the distribution of the corrections is reflected in the total time to rendezvous, where in Case VI the rendezvous occurred 179 seconds later than in Case V.

IMPLICATIONS OF RESULTS

From the results of this study, the courses of action that will be followed when τ_1 and τ_2 are used in conjunction with a guidance system can be catalogued. If the ferry is on a collision course, then the optimum time to intercept and the optimum time to rendezvous have the same value and hence the same course of action is called for, namely, "do nothing."

If the vehicle is not on a collision or intercept course, then the optimum times to intercept and rendezvous are different and so are the indicated courses of action. For the minimum (or very near minimum) velocity change to intercept $\left|\Delta V_1(\omega\tau_1)\right|$, the line-of-sight velocity $\dot{R}$ should remain unchanged while the in-plane angular velocity $R\dot{\theta}\cos\psi$ should be changed to put the vehicle on a collision course. Although not proven in this paper, it can be shown that the in-plane and out-of-plane motions are only loosely coupled and that velocity changes in the two planes may be calculated separately. However, the actual velocity changes should be made simultaneously (whenever possible) in order to profit from using the vector sum of these changes rather than the scalar sum, since

$$\sqrt{\Delta V_{x,y}{}^2 + \Delta V_z{}^2} \leqq \overline{\Delta V}_{x,y} + \overline{\Delta V}_z$$

If the ultimate objective is soft rendezvous, then the sum of the two velocity changes $|\Delta V_1|$ and $|\Delta V_2|$ should be minimized. If the time to rendezvous (or anomaly to rendezvous) which minimizes the sum of these velocity changes is found by some means such as equation (12), then a reduction in the closing velocity $\dot{R}_o$ will be required to satisfy that condition. At no time should the line-of-sight velocity ever be increased if near-minimum velocity for interception or rendezvous is desired.

For the terminal phase of rendezvous where the relative velocity of the two vehicles is brought to zero, it would appear that any guidance law based on first-order or uniform gravity fields will probably suffice with little or no penalty associated with nonoptimization so long as the line-of-sight velocity is always reduced and never increased.

When the time or anomaly until rendezvous is large, the degree of approximation of the guidance equations becomes important. Uniform gravity equations (and hence the expressions for τ_1 and τ_2 given by eqs. (8) and (12)) appear to be valid only out to an $\omega\tau$ of about 30° to 40°. The first-order gravity equations appear to be valid out to an $\omega\tau$ near 90°. Since the variation of $|\Delta V_1| + |\Delta V_2|$ is very nearly flat in the region of $\omega\tau > \omega\tau_{min}$, it would appear that overestimating the value of the minimum would not be very expensive in fuel, and might be desirable if the exact value of $\omega\tau_2$ were uncertain.

For guidance purposes, more accurate expressions for the minimums of the ΔV curves than those given by equations (8) and (12) would be desirable in order to avoid the necessity of computing the variations of ΔV_1 and ΔV_2 until a minimum is reached. However, all such solutions found to date have involved rather complicated transcendental expressions in $\omega\tau$, requiring iteration procedures or numerical solutions equal in complexity to the direct calculation of the ΔV variations with $\omega\tau$.

A more general formulation of the orbital mechanics method for rendezvous guidance would also be helpful. Many papers have been written on minimum energy transfer trajectories between two well-defined orbits. What is now needed is an extension of this problem to that of finding the optimum transfer trajectory from any given set of initial conditions to the final condition of rendezvous with a real or an imaginary body moving in a well-defined trajectory or orbit. Solutions to this problem, even the approximate ones given in this paper, have application not only to the earth rendezvous problem, but to the launch of space payloads, launch abort trajectories, lunar rendezvous, abort from lunar landings, and other similar problems.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Air Force Base, Va., December 18, 1961.

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APPENDIX

DERIVATION OF THE EXPRESSIONS FOR OPTIMUM TIME TO INTERCEPT AND RENDEZVOUS USING UNIFORM GRAVITY EQUATIONS

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Equations (4) and (9) were obtained by using the first-order gravity equations of Clohessy and Wiltshire marked as T-1 in table I. When these equations are used to calculate the optimum velocity change for minimum fuel consumption, extensive use of power series is required, which tends to obscure the results. Therefore, for the sake of clarity and simplicity, the uniform gravity equations of motion used by Hornby (marked T-2 in table I) will be used in this appendix to derive expressions for the optimum time to intercept and rendezvous. The validity of the results is discussed in the body of the paper.

The uniform gravity set of equations, namely,

$$\left.\begin{aligned} \ddot{x} - 2\omega\dot{y} &= \frac{T_x}{m} \\ \ddot{y} + 2\omega\dot{x} &= \frac{T_y}{m} \\ \ddot{z} + \omega^2 z &= \frac{T_z}{m} \end{aligned}\right\} \tag{A1}$$

have the homogeneous $(\overline{T} = 0)$ solutions for the components of velocity:

$$\begin{Bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{Bmatrix} = \begin{bmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ -\sin 2\omega t & \cos 2\omega t & 0 \\ 0 & 0 & \cos \omega t \end{bmatrix} \begin{Bmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{z}_o \end{Bmatrix} - \omega \sin \omega t \begin{Bmatrix} 0 \\ 0 \\ z_o \end{Bmatrix} \tag{A2}$$

and displacement:

$$\begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix} = \frac{1}{\omega} \begin{bmatrix} \sin \omega t \cos \omega t & \sin^2 \omega t & 0 \\ -\sin^2 \omega t & \sin \omega t \cos \omega t & 0 \\ 0 & 0 & \sin \omega t \end{bmatrix} \begin{Bmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{z}_o \end{Bmatrix} + \begin{Bmatrix} x_o \\ y_o \\ z_o \cos \omega t \end{Bmatrix} \tag{A3}$$

In order to achieve collision, it is required that $x = y = z = 0$ when t is equal to some specified value τ. Setting this condition into equation (A3) and solving for $\dot{x}_o$, $\dot{y}_o$, $\dot{z}_o$ yields

$$\overline{V}_1(\tau) = \begin{Bmatrix} \dot{x}_o'(\tau) \\ \dot{y}_o'(\tau) \\ \dot{z}_o'(\tau) \end{Bmatrix} = -\omega \begin{bmatrix} \cot \omega\tau & -1 & 0 \\ 1 & \cot \omega\tau & 0 \\ 0 & 0 & \cot \omega\tau \end{bmatrix} \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix} \tag{A4}$$

Primes have been used to indicate that these are the calculated velocity components required to achieve interception at time τ in the future.

If $\overline{V}_o$ is the present velocity relative to the target and $\overline{V}_1(\tau)$ is the velocity required to intercept at $t = \tau$, then the indicated velocity change is

$$\Delta\overline{V}_1(\tau) = \overline{V}_1(\tau) - \overline{V}_o$$

where

$$\overline{V}_1 = (\dot{x}_o', \quad \dot{y}_o', \quad \dot{z}_o')$$

$$\overline{V}_o = (\dot{x}_o, \quad \dot{y}_o, \quad \dot{z}_o)$$

If this velocity correction is made, then the new velocity (as a function of time) is obtained by substituting equation (A4) into equation (A2) ($\dot{x}_o'$ is substituted for $\dot{x}_o$, etc.):

$$\begin{Bmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \\ \dot{z}_1(t) \end{Bmatrix} = -\omega \begin{bmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ -\sin 2\omega t & \cos 2\omega t & 0 \\ 0 & 0 & \cos \omega t \end{bmatrix} \begin{bmatrix} \cot \omega\tau & -1 & 0 \\ 1 & \cot \omega\tau & 0 \\ 0 & 0 & \cot \omega\tau \end{bmatrix} \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix}$$

$$- \omega \sin \omega t \begin{Bmatrix} 0 \\ 0 \\ z_o \end{Bmatrix} \tag{A5}$$

The subscript 1 has been used to indicate the components of this new velocity. From equation (A5) the velocity at the time of interception τ can be calculated by setting $t = \tau$:

$$\overline{V}_2(\tau) = \begin{Bmatrix} \dot{x}_1(\tau) \\ \dot{y}_1(\tau) \\ \dot{z}_1(\tau) \end{Bmatrix} = -\omega \begin{bmatrix} \cot \omega\tau & 1 & 0 \\ -1 & \cot \omega\tau & 0 \\ 0 & 0 & \csc \omega\tau \end{bmatrix} \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix} \tag{A6}$$

At interception this relative velocity must be brought to zero so that the required velocity change is

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$$\Delta\overline{V}_2(\tau) = \overline{V}_3 - \overline{V}_2(\tau)$$

where $\overline{V}_3 = (0,0,0)$.

In theory then, rendezvous can be achieved with two velocity changes. The magnitudes of these velocity changes are

$$\left|\Delta V_1(\omega\tau)\right| = \sqrt{\left[\dot{x}_o - \dot{x}_o'(\tau)\right]^2 + \left[\dot{y}_o - \dot{y}_o'(\tau)\right]^2 + \left[\dot{z}_o - \dot{z}_o'(\tau)\right]^2}$$

$$= \left[\omega^2\left(x_o^2 + y_o^2 + z_o^2\right)\cot^2\omega\tau + 2\omega(x_o\dot{x}_o + y_o\dot{y}_o\right.$$

$$\left. + z_o\dot{z}_o)\cot \omega\tau + (\omega y_o - \dot{x}_o)^2 + (\dot{y}_o + \omega x_o)^2 + \dot{z}_o\right]^{1/2} \tag{A7}$$

$$\left|\Delta V_2(\omega\tau)\right| = \sqrt{\left[0 - \dot{x}_1(\tau)\right]^2 + \left[0 - \dot{y}_1(\tau)\right]^2 + \left[0 - \dot{z}_1(\tau)\right]^2}$$

$$= \omega\sqrt{x_o^2 + y_o^2 + z_o^2} \csc \omega\tau \tag{A8}$$

From this point on it is more convenient to use spherical rather than rectangular coordinates. (See fig. 2(a).) Accordingly, define

$$R_o = \sqrt{x_o^2 + y_o^2 + z_o^2}$$

$$\theta = \tan^{-1}\left(\frac{-y}{x}\right)$$

$$\psi = \tan^{-1}\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$

In terms of these coordinates, equations (A7) and (A8) become

$$\left|\Delta V_1(\omega\tau)\right| = \sqrt{\left[\omega R_o \cot \omega\tau + \dot{R}_o\right]^2 + K} \tag{A9}$$

$$\left|\Delta V_2(\omega\tau)\right| = \omega R_o \csc \omega\tau \tag{A10}$$

where

$$K = R_o^2\left[\dot{\psi}_o^{\,2} + \cos^2\psi_o(\dot{\theta}_o - \omega)^2\right]$$

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It should be noted that K is not a function of $\omega\tau$.

The minimum velocity change required for an interception course (hence, the optimum time to intercept) is obtained when

$$\frac{\partial\left|\Delta V_1\right|}{\partial(\omega\tau)} = 0$$

This condition is satisfied when

$$\omega R_o \cot \omega\tau + \dot{R}_o = 0$$

or

$$\frac{\omega R_o}{-\dot{R}_o} = \tan \omega\tau = \omega\tau + \frac{(\omega\tau)^3}{3} + \frac{(\omega\tau)^5}{5} + \ . \ . \ .$$

$$\approx \omega\tau = \text{(by definition)}\ \ \omega\tau_1 \tag{A11}$$

This is the same approximate result obtained in reference 1 by using first-order gravity equations (eqs. T-1 of table I).

The minimum total velocity change required to rendezvous (optimum time to rendezvous) is obtained when

$$\frac{\partial}{\partial(\omega\tau)}\left(\left|\Delta V_1\right| + \left|\Delta V_2\right|\right) = 0$$

This condition is satisfied when

$$-\cos \omega\tau = \frac{\omega R_o \cot \omega\tau + \dot{R}_o}{\left[(\omega R_o \cot \omega\tau + \dot{R}_o)^2 + K\right]^{1/2}}$$

By squaring, rearranging, and noting that

$$\frac{\cot^2\omega\tau}{\cot^2\omega\tau + 1} = \cos^2\omega\tau = \frac{\left(\frac{\omega R_o \cot \omega\tau + \dot{R}_o}{\sqrt{K}}\right)^2}{\left(\frac{\omega R_o \cot \omega\tau + \dot{R}_o}{\sqrt{K}}\right)^2 + 1}$$

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it can be seen that

$$\pm\cot \omega\tau = \frac{\omega R_o \cot \omega\tau + \dot{R}_o}{\sqrt{K}}$$

is a solution. This expression can be solved for $\omega\tau$ such that

$$\frac{\omega R_o \pm \sqrt{K}}{-\dot{R}_o} = \tan \omega\tau$$

$$\approx \omega\tau = \text{(by definition)} \ \omega\tau_2 \tag{A12}$$

Equations (A11) and (A12) thus give the expressions used for the optimum time to intercept τ_1 and rendezvous τ_2, respectively. Putting in the expression for K yields

$$\tau_1 = \frac{R_o}{-\dot{R}_o} \tag{A13}$$

$$\tau_2 = \frac{R_o}{-\dot{R}_o} \pm \sqrt{\frac{R_o^{\,2}\left[\dot{\psi}_o^{\,2} + (\dot{\theta}_o - \omega)^2\cos^2\psi_o\right]}{\omega^2\dot{R}_o^{\,2}}} \tag{A14}$$

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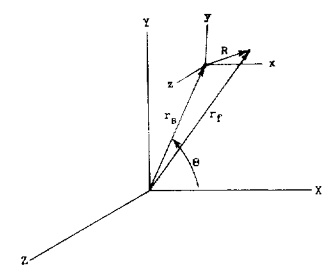
TABLE I.- ORBITAL MECHANICS

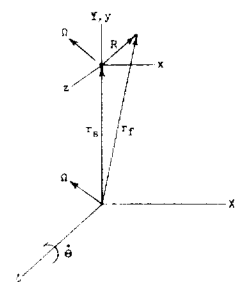
Inertially fixed axes



Assumptions	Vector form	Rectangular coordinates
Exact	$\frac{\partial^2 \bar{r}_f}{\partial t^2} - \nabla\left(\frac{\mu}{r_f}\right) = \frac{\bar{T}}{m}$	Similar to form immediately below except in the expansion of $\nabla\left(\frac{\mu}{r_f}\right)$
Spherical earth	$\frac{d^2 \bar{r}_f}{dt^2} + \frac{GM}{r_f^3}\bar{r}_f = \frac{\bar{T}}{m}$ $\bar{r}_f = \bar{r}_s + \bar{R}$ $\bar{r}_s = (X, Y, Z)$ $= (r_s \cos\theta, r_s \sin\theta, 0)$ $\bar{R} = (x, y, z)$	$\ddot{x} + \left(\ddot{r}_s - r_s\dot{\theta}^2\right)\cos\theta - \left(2\dot{r}_s\dot{\theta} + r_s\ddot{\theta}\right)\sin\theta + \frac{GM}{r_f^3}(x + r_s\cos\theta) = \frac{T_x}{m}$ $\ddot{y} + \left(\ddot{r}_s - r_s\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}_s\dot{\theta} + r_s\ddot{\theta}\right)\cos\theta + \frac{GM}{r_f^3}(y + r_s\sin\theta) = \frac{T_y}{m}$ $\ddot{z} + \frac{GM}{r_f^3}z = \frac{T_z}{m}$
Spherical earth, Station in a circular orbit	$\frac{d^2\bar{R}}{dt^2} + \frac{d^2\bar{r}_s}{dt^2} + \frac{GM}{r_f^3}\bar{r}_f = \frac{\bar{T}}{m}$	$\ddot{x} - r_s\omega^2\cos\theta + \frac{GM}{r_f^3}(x + r_s\cos\theta) = \frac{T_x}{m}$ $\ddot{y} - r_s\omega^2\sin\theta + \frac{GM}{r_f^3}(y + r_s\sin\theta) = \frac{T_y}{m}$ $\ddot{z} + \frac{GM}{r_f^3}z = \frac{T_z}{m}$
Spherical earth, Circular orbit, 1st order gravity field	$\frac{d^2\bar{R}}{dt^2} + \frac{GM}{r_s^3}\left(\bar{R} - 3\,\frac{\bar{r}_s \cdot \bar{R}}{r_s^2}\,\bar{r}_s\right) = \frac{\bar{T}}{m}$ refs. 3, 6, and 7	$\ddot{x} + \omega^2(x - 3x\cos^2\theta - 3y\sin\theta\cos\theta) = \frac{T_x}{m}$ $\ddot{y} + \omega^2(y - 3x\sin\theta\cos\theta - 3y\sin^2\theta) = \frac{T_y}{m}$ $\ddot{z} + \omega^2 z = \frac{T_z}{m}$
Spherical earth, Circular orbit, Uniform gravity field		
No gravity	$\frac{d^2\bar{R}}{dt^2} = \frac{\bar{T}}{m}$ any orbit ref. 6	Circular orbit $\ddot{x} = \frac{T_x}{m}$ $\ddot{y} = \frac{T_y}{m}$ $\ddot{z} = \frac{T_z}{m}$

EQUATIONS SUITABLE FOR RENDEZVOUS GUIDANCE

Rotating set of axes



Comments	Vector form	Rectangular coordinates
$\bar r_f = \bar r_s + \bar R$ $\frac{\mu}{r_f}$ is the gravity potential due to earth, moon, planets, etc.	$\frac{\partial^2 \bar r_f}{\partial t^2} + 2\,\bar\Omega\times\frac{\partial \bar r_f}{\partial t} + \dot{\bar\Omega}\times\bar r_f + \bar\Omega\times\bar\Omega\times\bar r_f - \nabla\left(\frac{\mu}{r_f}\right) = \frac{\bar T}{m}$	Similar to form immediately below except in the expansion of $\nabla\left(\frac{\mu}{r_f}\right)$
$\dot\theta$ is the angular velocity of station about center of earth r_s is radial position of station	$\frac{d^2\bar r_f}{dt^2} + 2\,\bar\Omega\times\frac{d\bar r_f}{dt} + \dot{\bar\Omega}\times\bar r_f + \bar\Omega\times\bar\Omega\times\bar r_f + \frac{GM}{r_f^3}\,\bar r_f = \frac{\bar T}{m}$ $\bar r_f = (x,\ y + r_s,\ z) = \bar r_s + \bar R$ $\bar r_s = (0,\ r_s,\ 0)$ $\bar R = (x,\ y,\ z)$ $\bar\Omega = (0,\ 0,\ \dot\theta)$	$\ddot x - (y + r_s)\ddot\theta - 2(\dot y + \dot r_s)\dot\theta - x\left(\dot\theta^2 - \frac{GM}{r_f^3}\right) = \frac{T_x}{m}$ $\ddot y + x\ddot\theta + 2\dot x\dot\theta + \ddot r_s - (y + r_s)\left(\dot\theta^2 - \frac{GM}{r_f^3}\right) = \frac{T_y}{m}$ $\ddot z + \frac{GM}{r_f^3}\,z = \frac{T_z}{m}$ refs. 4 and 5
$\bar r_s(\theta) \neq$ Constant r_s = Constant $\dot\theta$ = Constant = ω $\bar\Omega$ = Constant = $(0,\ 0,\ \omega) = \bar\omega$ $\omega^2 = \frac{GM}{r_s^3} = \dot\theta^2$	$\frac{d^2\bar R}{dt^2} + 2\,\bar\omega\times\frac{d\bar R}{dt} + \bar\omega\times\bar\omega\times\bar r_f + \frac{GM}{r_f^3}\,\bar r_f = \frac{\bar T}{m}$ or $\frac{d^2\bar R}{dt^2} + 2\,\bar\omega\times\frac{d\bar R}{dt} + \omega^2 z\bar k + \left(\frac{GM}{r_f^3} - \omega^2\right)\bar r_f = \frac{\bar T}{m}$	$\ddot x - 2\omega\dot y - x\left(\omega^2 - \frac{GM}{r_f^3}\right) = \frac{T_x}{m}$ $\ddot y + 2\omega\dot x - (y + r_s)\left(\omega^2 - \frac{GM}{r_f^3}\right) = \frac{T_y}{m}$ (T-3) $\ddot z + \frac{GM}{r_f^3}\,z = \frac{T_z}{m}$ refs. 4 and 5
$\frac{GM}{r_f^3} = \frac{GM}{r_s^3}\left(1 - 3\,\frac{\bar r_s\cdot\bar R}{r_s^2} + \dots\right)$ $\frac{d^2\bar r_s}{dt^2} = -\omega^2\bar r_s$ On the left $\frac{d^2\bar r_s}{dt^2} + \frac{GM}{r_f^3}\,\bar r_f \approx \frac{GM}{r_s^3}\left(\bar R - 3\,\frac{\bar r_s\cdot\bar R}{r_s^2}\,\bar r_s\right)$ On the right $\left(\frac{GM}{r_f^3} - \omega^2\right)\bar r_f \approx \frac{GM}{r_s^3}\left(-3\,\frac{y}{r_s}\,\bar r_s\right)$	$\frac{d^2\bar R}{dt^2} + 2\,\bar\omega\times\frac{d\bar R}{dt} + \bar\omega\times\bar\omega\times\bar R + \frac{GM}{r_s^3}(\bar R - \bar j3y) = \frac{\bar T}{m}$ or $\frac{d^2\bar R}{dt^2} + 2\,\bar\omega\times\frac{d\bar R}{dt} + \frac{GM}{r_s^3}(-\bar j3y + \bar kz) = \frac{\bar T}{m}$	$\ddot x - 2\omega\dot y = \frac{T_x}{m}$ $\ddot y + 2\omega\dot x - 3\omega^2 y = \frac{T_y}{m}$ (T-1) $\ddot z + \omega^2 z = \frac{T_z}{m}$ refs. 2, 4, 5 and 8 to 10
$\frac{GM}{r_f^3}\,\bar r_f \approx \frac{GM}{r_s^3}(\bar r_s + \bar R)$ $\frac{GM}{r_s^3} = \omega^2$	$\frac{d^2\bar R}{dt^2} + 2\,\bar\omega\times\frac{d\bar R}{dt} + \bar\omega\times\bar\omega\times\bar R + \frac{GM}{r_s^3}\,\bar R = \frac{\bar T}{m}$ or $\frac{d^2\bar R}{dt^2} + 2\,\bar\omega\times\frac{d\bar R}{dt} + \frac{GM}{r_s^3}\,\bar kz = \frac{\bar T}{m}$	$\ddot x - 2\omega\dot y = \frac{T_x}{m}$ $\ddot y + 2\omega\dot x = \frac{T_y}{m}$ (T-2) $\ddot z + \omega^2 z = \frac{T_z}{m}$
	$\frac{d^2\bar r_f}{dt^2} + 2\,\bar\Omega\times\frac{d\bar r_f}{dt} + \dot{\bar\Omega}\times\bar r_f + \bar\Omega\times\bar\Omega\times\bar r_f = \frac{\bar T}{m}$ for any orbit $\frac{d^2\bar R}{dt^2} + 2\,\bar\omega\times\frac{d\bar R}{dt} + \bar\omega\times\bar\omega\times\bar r_f = \frac{\bar T}{m}$ for circular orbit	Circular orbit $\ddot x - 2\omega\dot y - \omega^2 x = \frac{T_x}{m}$ $\ddot y + 2\omega x - \omega^2(y + r_s) = \frac{T_y}{m}$ $\ddot z = \frac{T_z}{m}$

TABLE II.- INITIAL CONDITIONS FOR THREE INTERCEPT TRAJECTORIES

[r_s = 4,260 statute miles; ω = 0.0011122 radian/sec]

L
1
8
9
5

Condition	Case I ($\omega\tau$ = 15.804°)	Case II ($\omega\tau$ = 32.118°)	Case III ($\omega\tau$ = 64.745°)
Rectangular coordinates			
x_o, statute miles	-32.18	-38.13	65.54
y_o, statute miles	-34.34	-87.37	-248.50
z_o, statute miles	0	-23.40	0
$\dot{x}_o$, ft/sec	462.3	-192.7	-2,053.0
$\dot{y}_o$, ft/sec	952.9	1,301.1	1,846.4
$\dot{z}_o$, ft/sec	0	217.8	0
Spherical coordinates			
R_o, statute miles	47.06	98.15	257.00
θ_o, deg	133.14	113.575	75.225
ψ_o, deg	0	193.793	0
$\dot{R}_o$, ft/sec	-1,011.44	-1,135.18	-2.308.86
$R_o\dot{\psi}_o$, ft/sec	0	-54.05	0
$R_o(\cos\psi_o)\dot{\theta}_o$, ft/sec . .	314.257	697.12	1,514.23

TABLE III.- CONDITIONS AND VELOCITY REQUIREMENTS FOR

MULTICORRECTION INTERCEPTION AND RENDEZVOUS

(a) Initial conditions

Condition	All cases		
$\omega\tau$ (estimated), deg	81.06		
x_o, statute miles	113.35		
y_o, statute miles	-202.66		
$\dot{x}_o$, ft/sec	-1,843.9		
$\dot{y}_o$, ft/sec	1,131.9		
	Case IV	Case V	Case VI
z_o, statute miles	0	0	50.0
$\dot{z}_o$, ft/sec	0	0	0
R_o, statute miles	232.21	232.21	237.53
$\dot{R}_o$, ft/sec	-1,888.03	-1,888.03	-1,845.67
In-plane guidance time	τ_1 (eq. (8))	τ_2 (eq. (13))	τ_2 (eq. (12))

(b) ΔV, t, and ωt required for rendezvous

	Case IV	Case V	Case VI
$\sum \Delta V_{x,y}$, ft/sec	164.98	350.22	490.18
+ ΔV_2, ft/sec	+ 687.20	+ 380.00	+ 235.83
Subtotal, ft/sec	852.18	730.22	726.01
+ $\sum \Delta V_z$, ft/sec	+ 0	+ 0	+ 328.24
Total, ft/sec	852.18	730.22	1,054.25
t, sec	1,231.09	1,407.72	1,586.62
ωt, deg	78.45	89.71	101.11

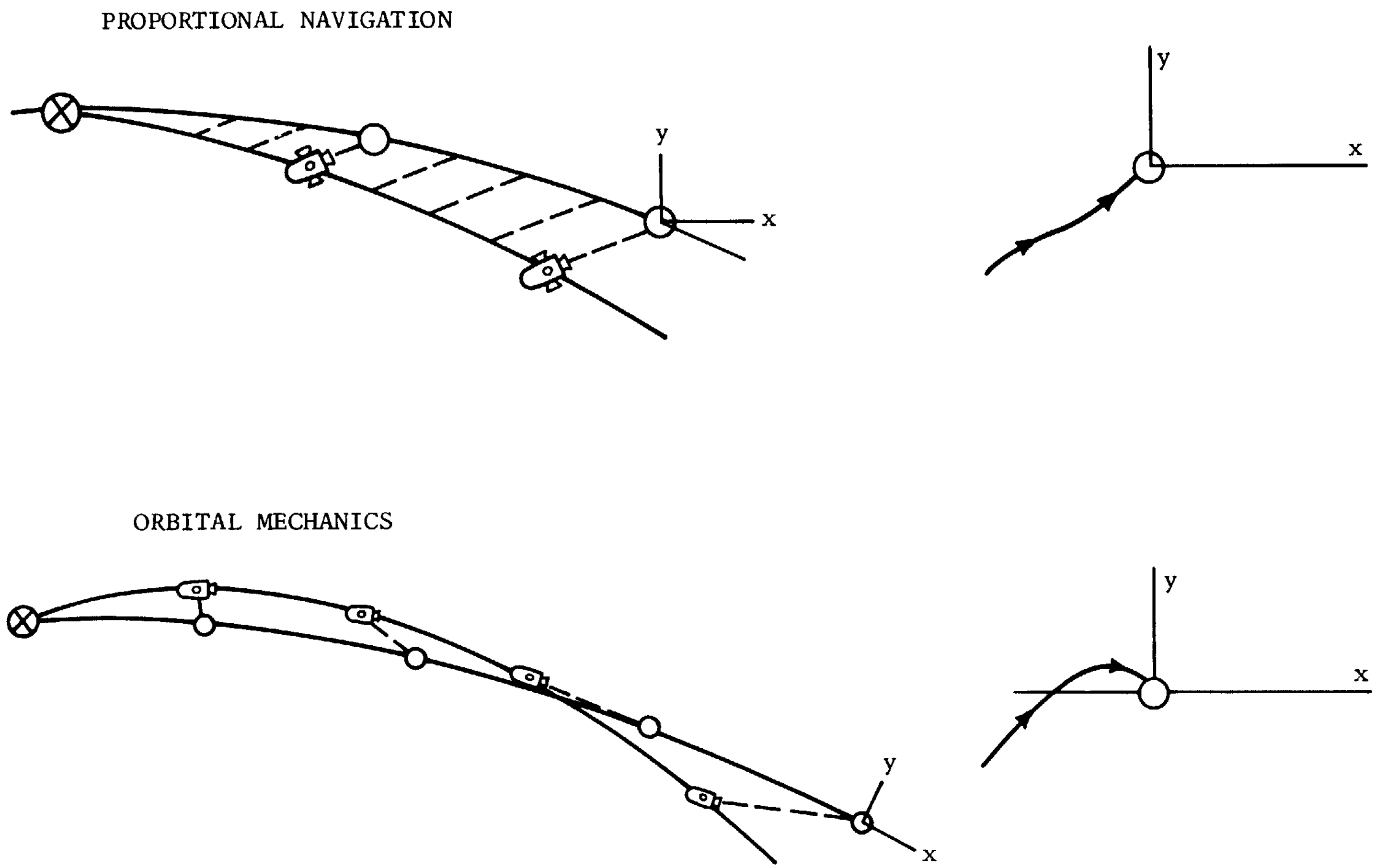
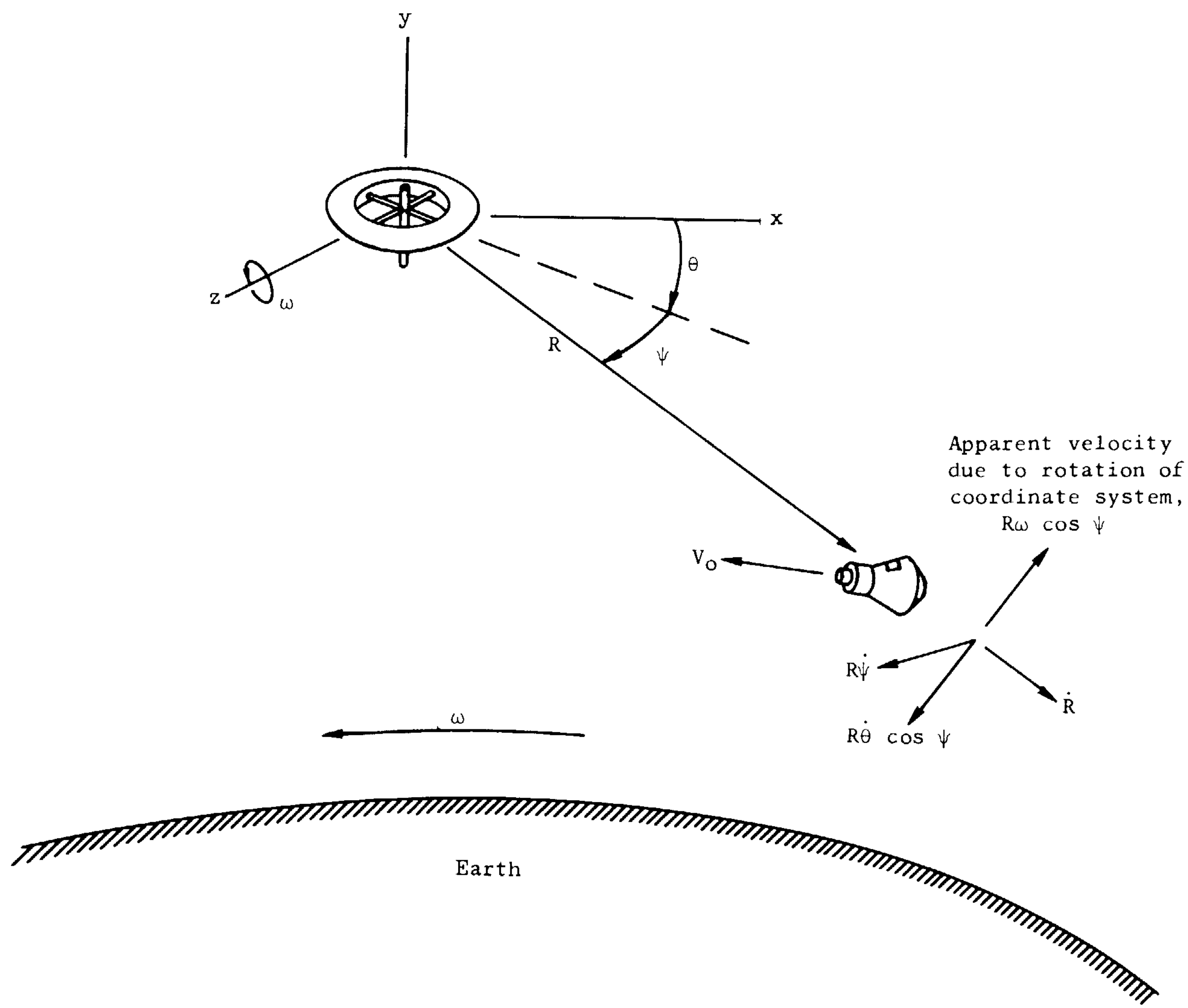
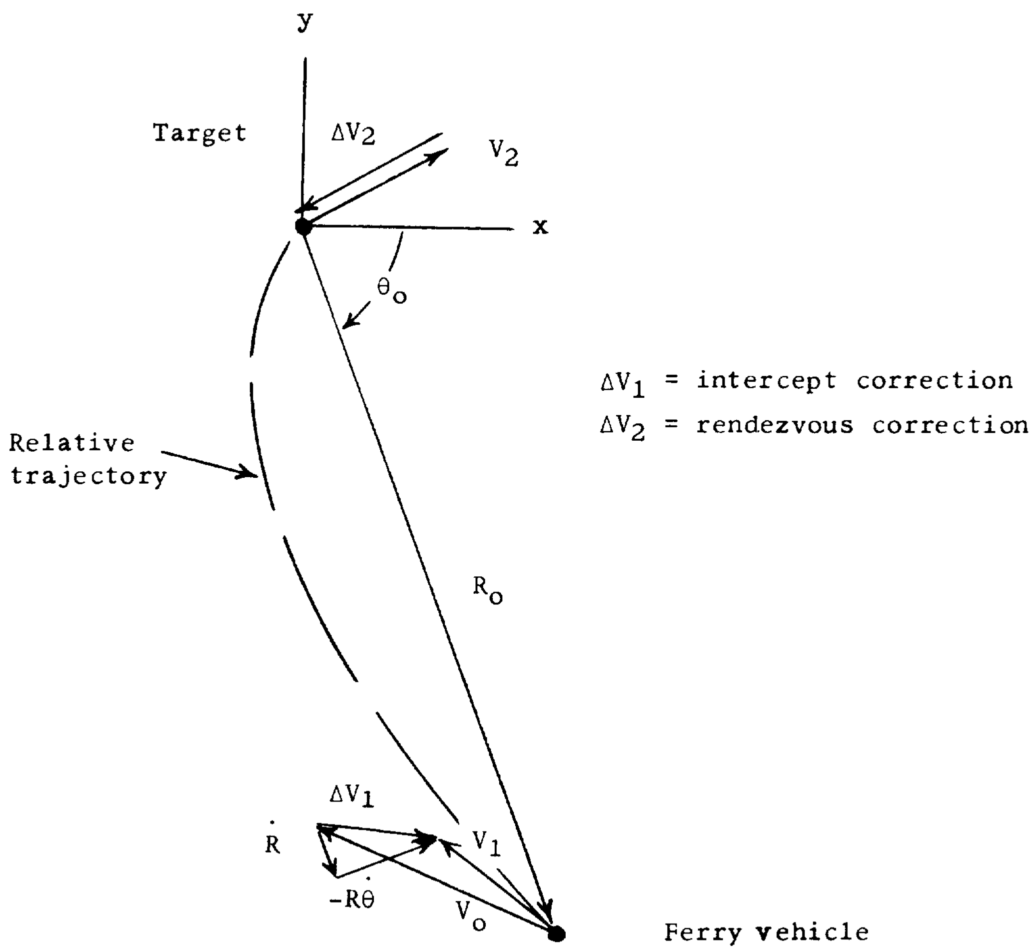


Figure 1.- Illustration of rendezvous techniques with guidance based on proportional navigation and orbital mechanics.



(a) Relative coordinates.

Figure 2.- Schematic drawing of coordinates and velocity changes for a two-impulse rendezvous.



(b) Velocity changes.

Figure 2.- Concluded.

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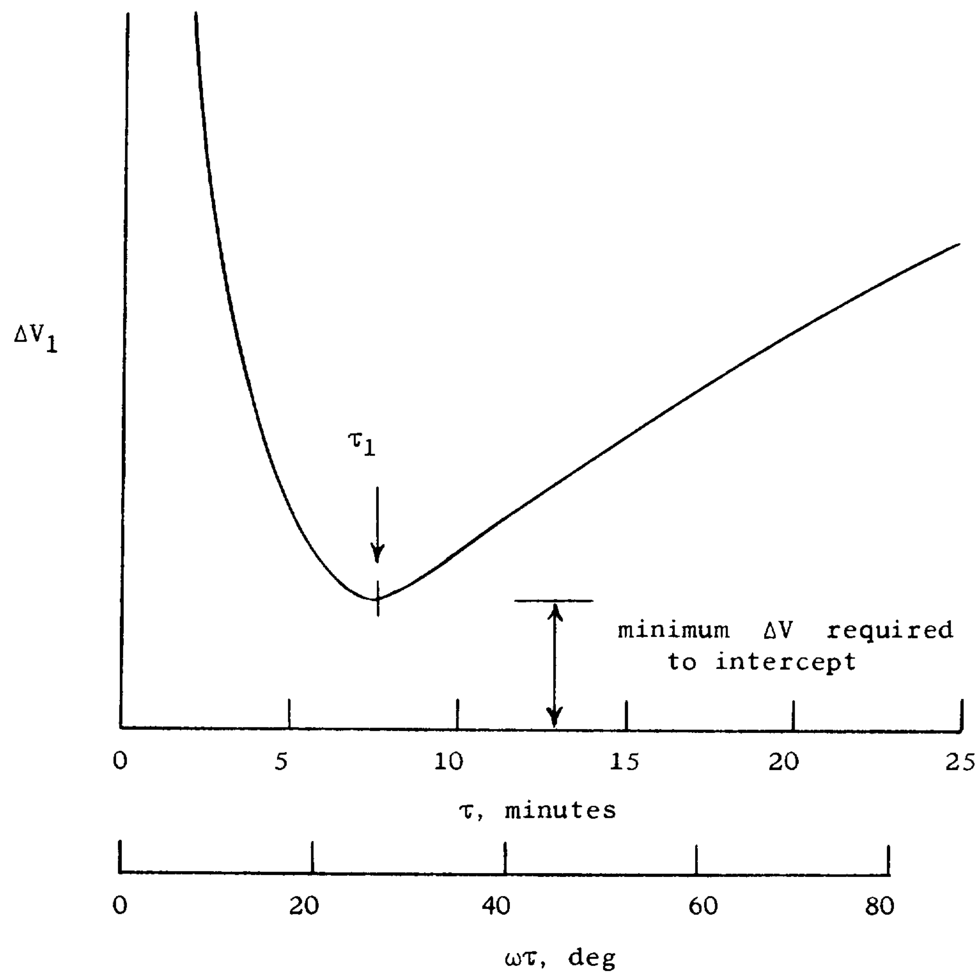
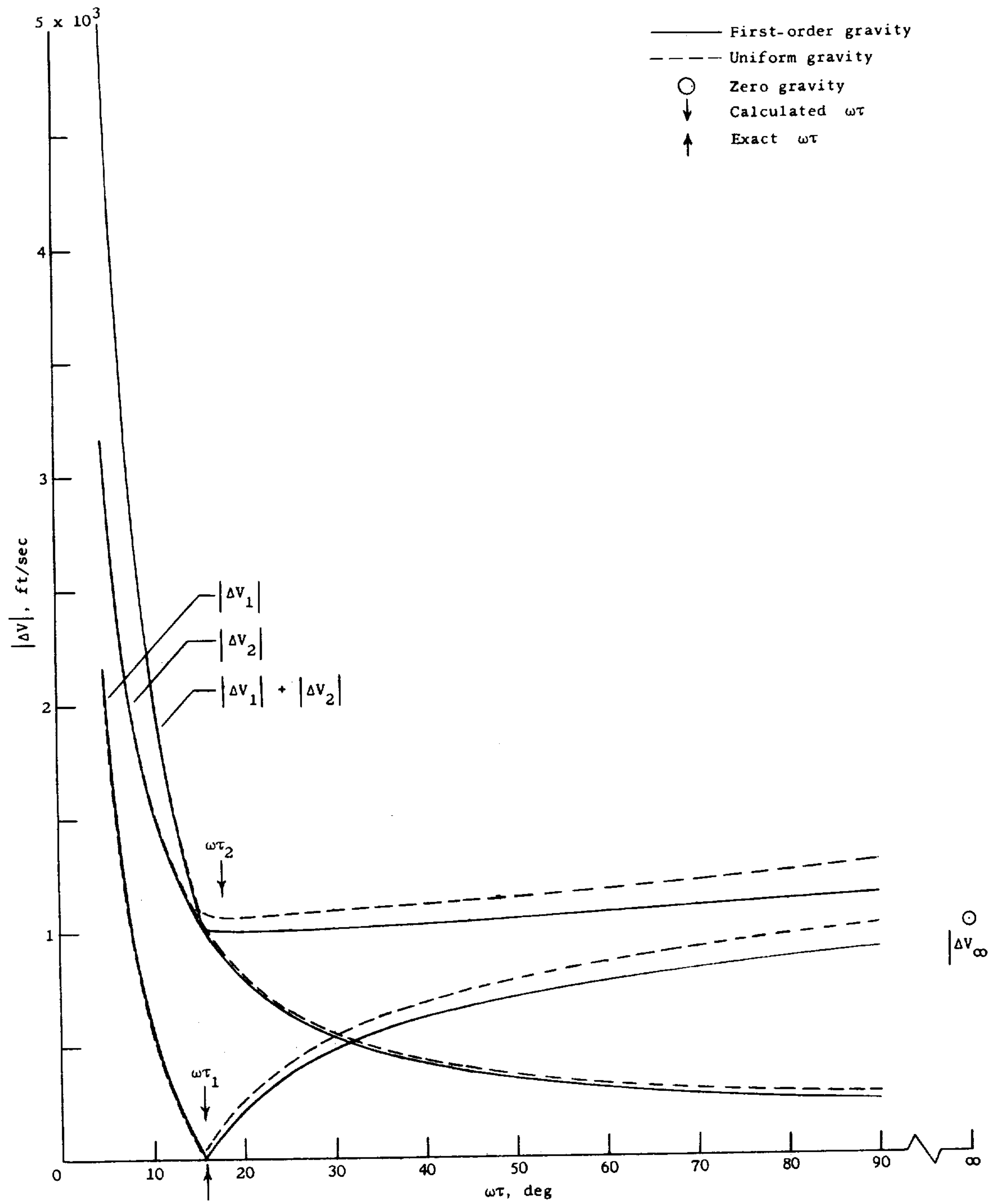


Figure 3.- Variation of velocity change required to intercept ΔV_1 with time τ and anomaly $\omega\tau$ to intercept.



(a) Case I, $\omega\tau = 15.8^{\circ}$.

Figure 4.- Variation of velocity change for intercept and rendezvous for the three sets of initial conditions given in table II.

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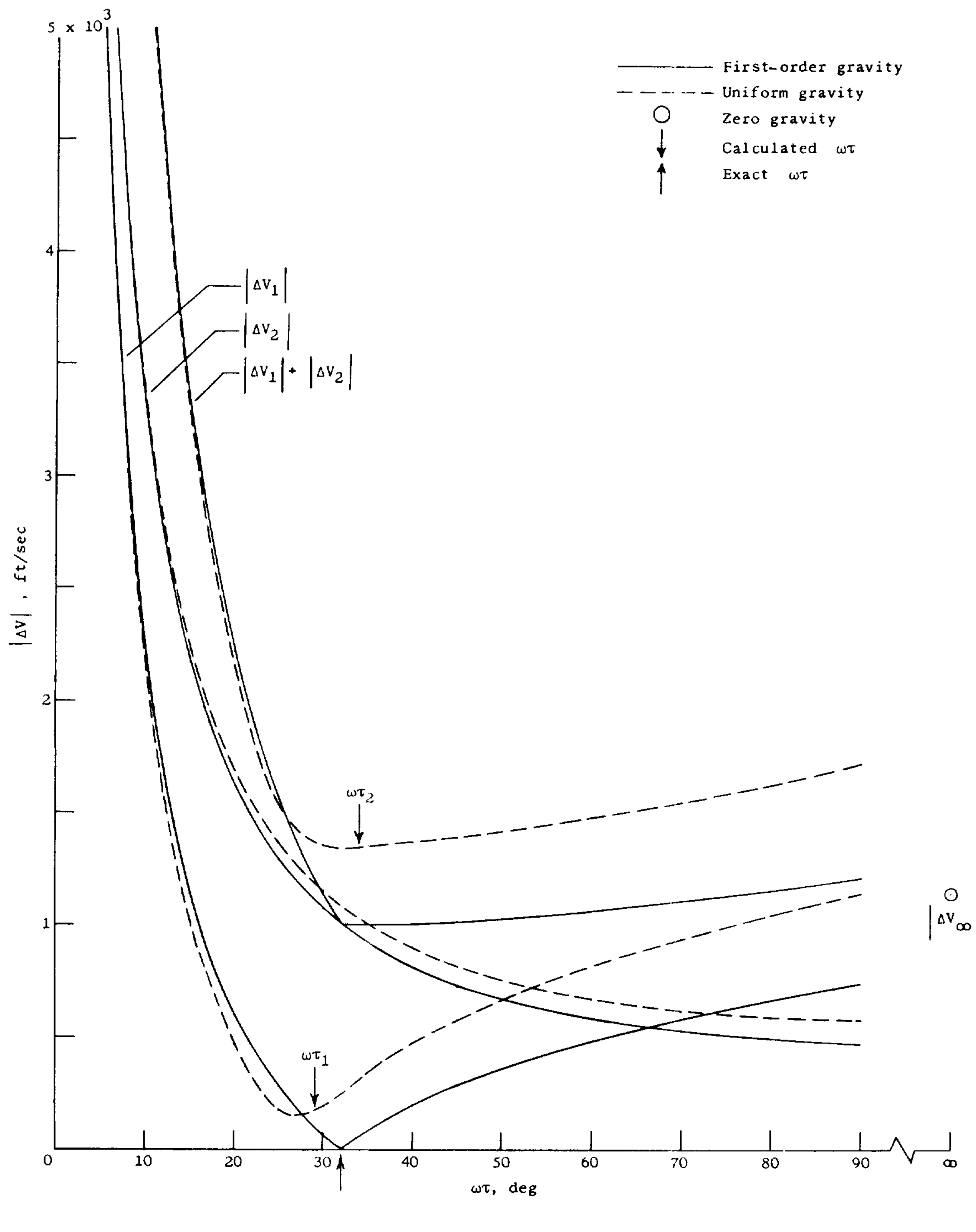
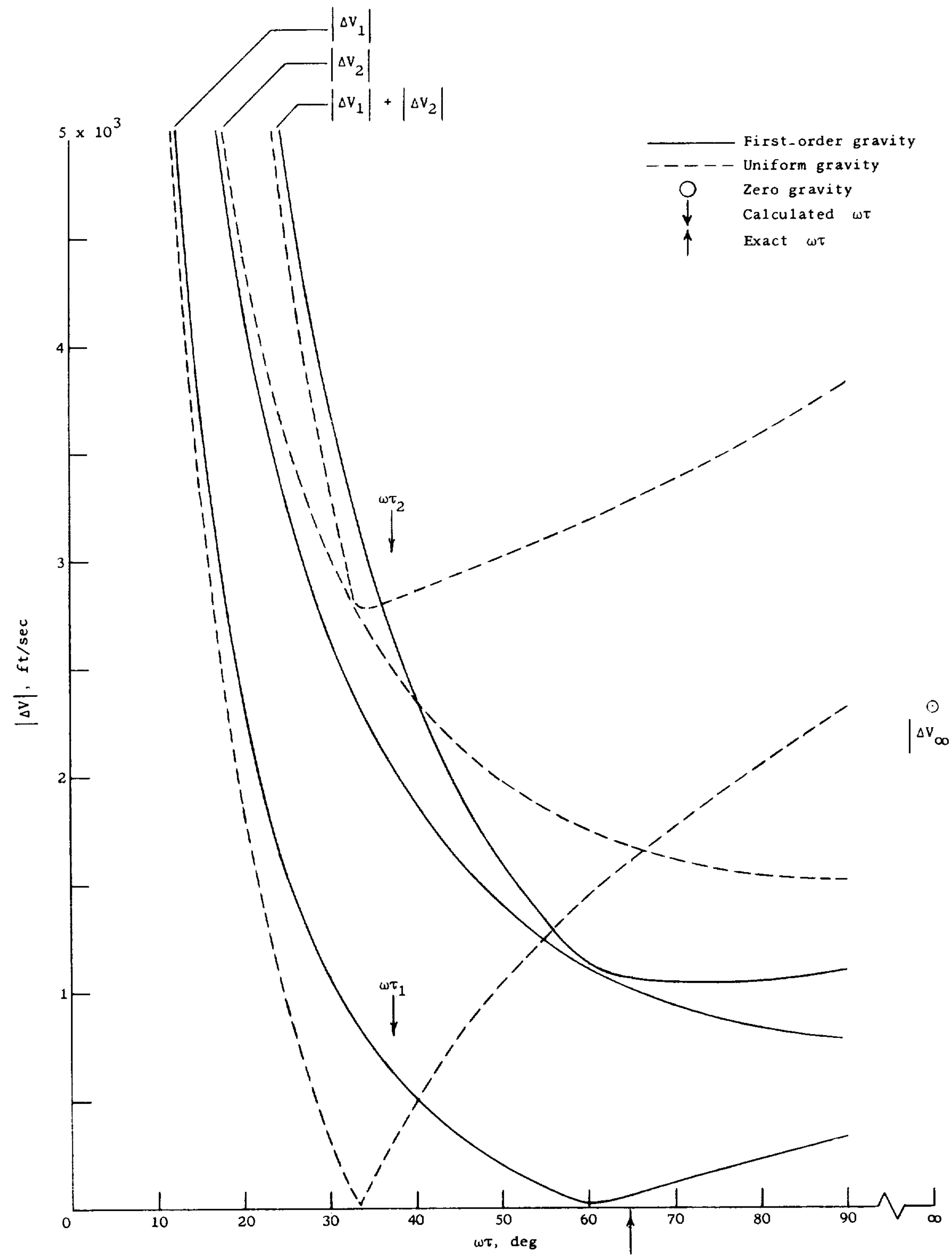


(b) Case II, $\omega\tau = 32.1^{\circ}$.

Figure 4.- Continued.



(c) Case III, $\omega\tau = 64.7^\circ$.

Figure 4.- Concluded.

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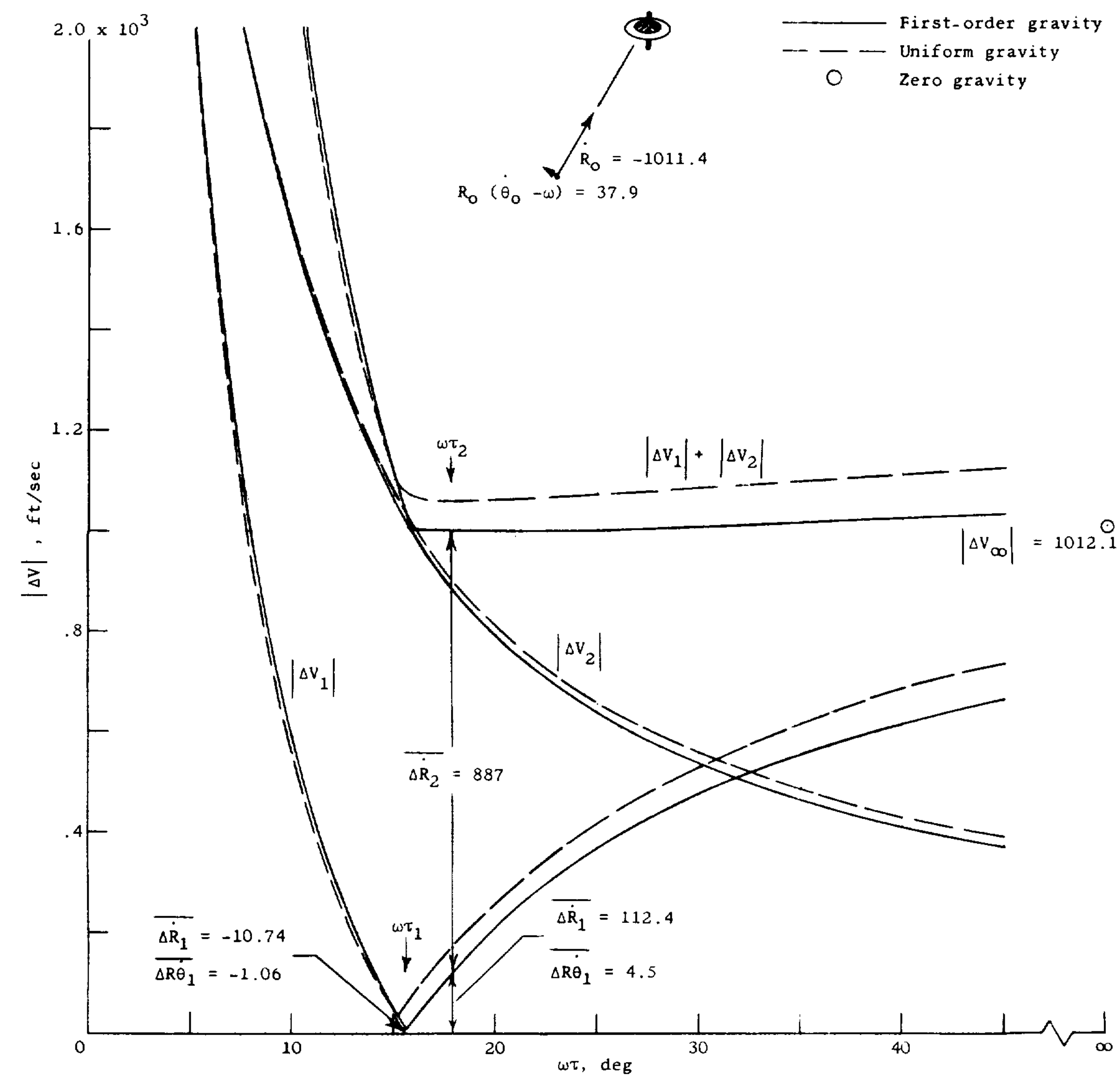

(a) Oncourse.

Figure 5.- Variation of velocity change for intercept and rendezvous for one oncourse and two offcourse initial conditions. All velocities and velocity changes are in ft/sec.

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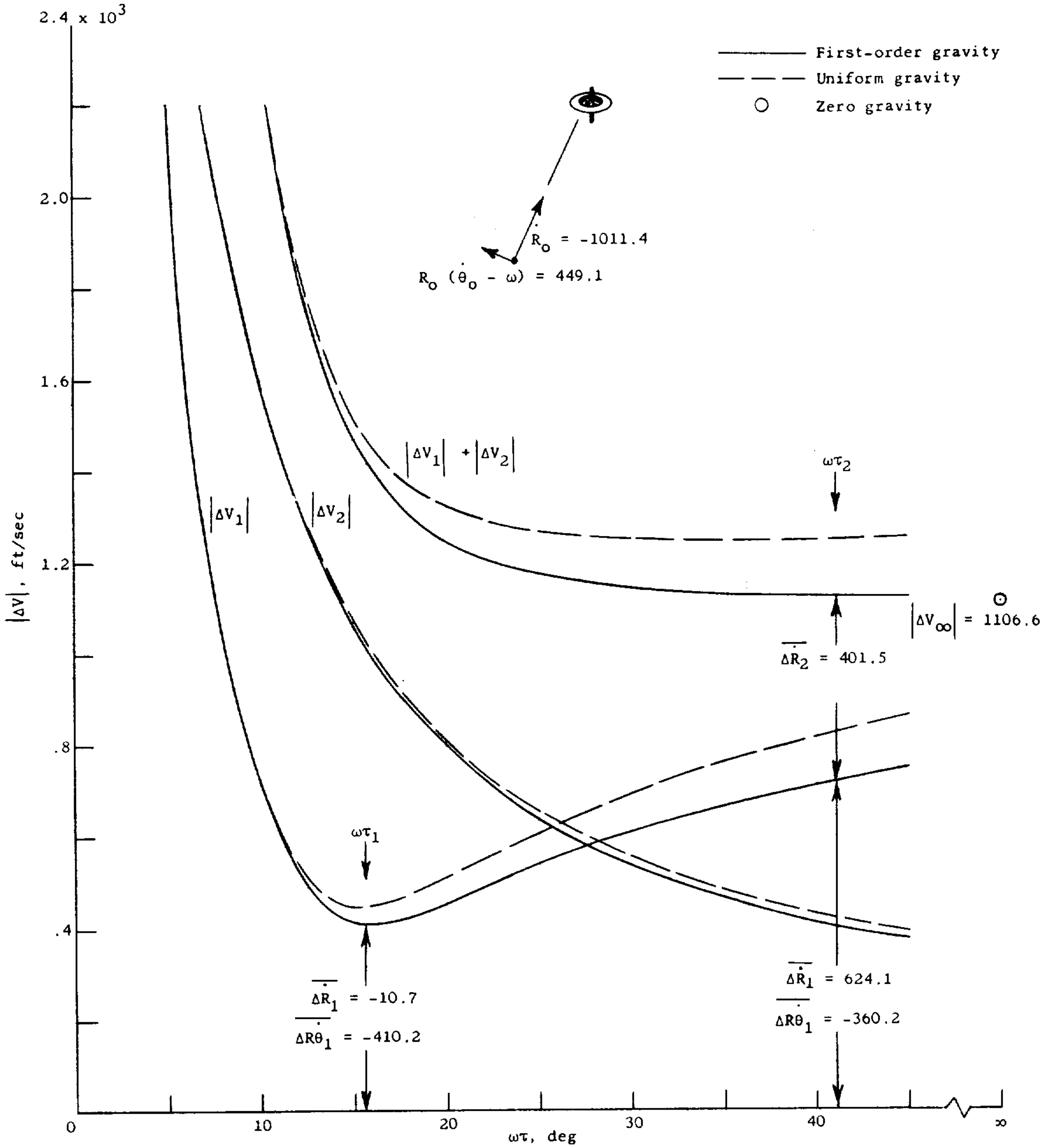

(b) Offcourse; $R_o\dot{\theta}_o$ increased by 411.2 ft/sec.

Figure 5.- Continued.

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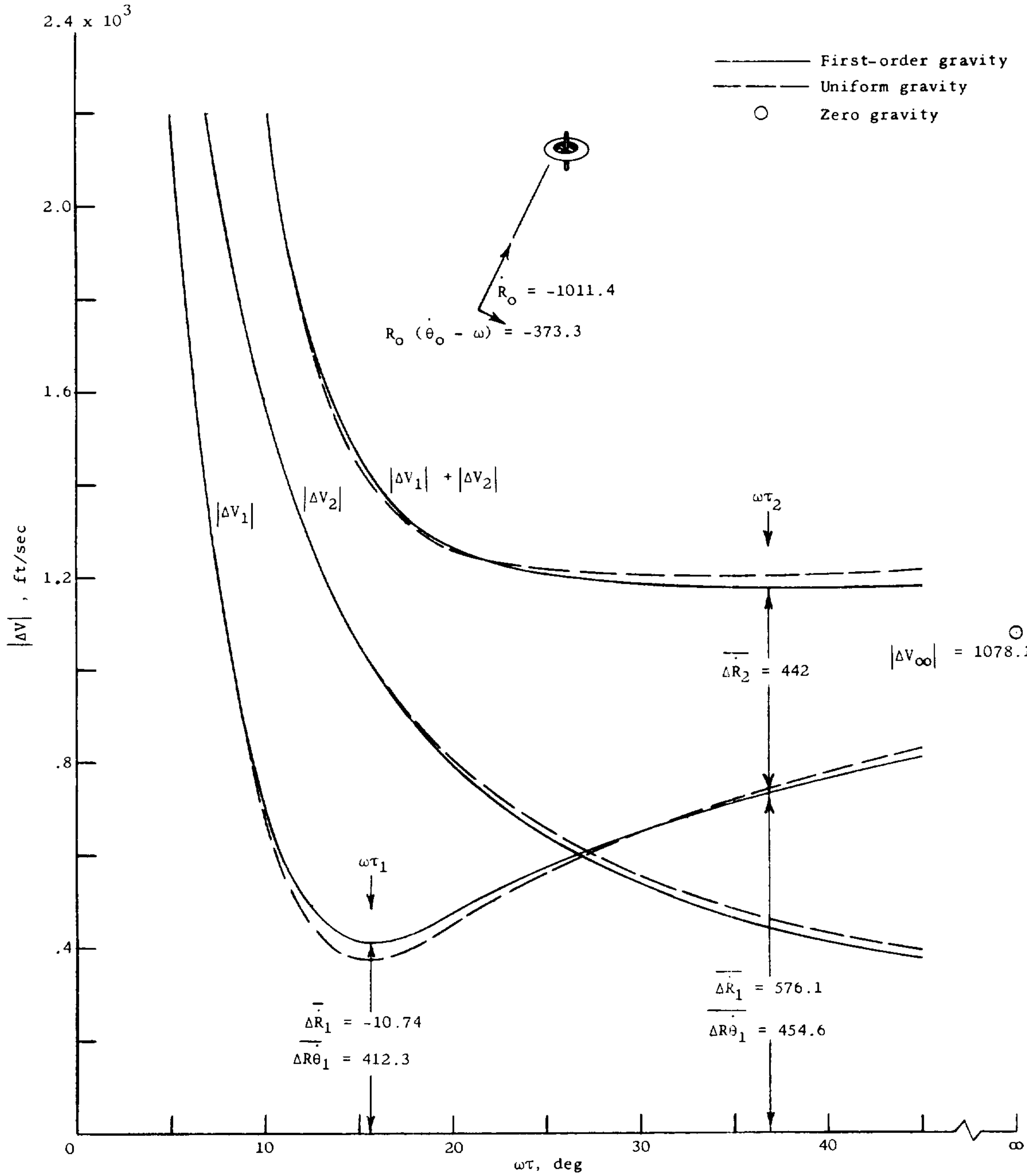
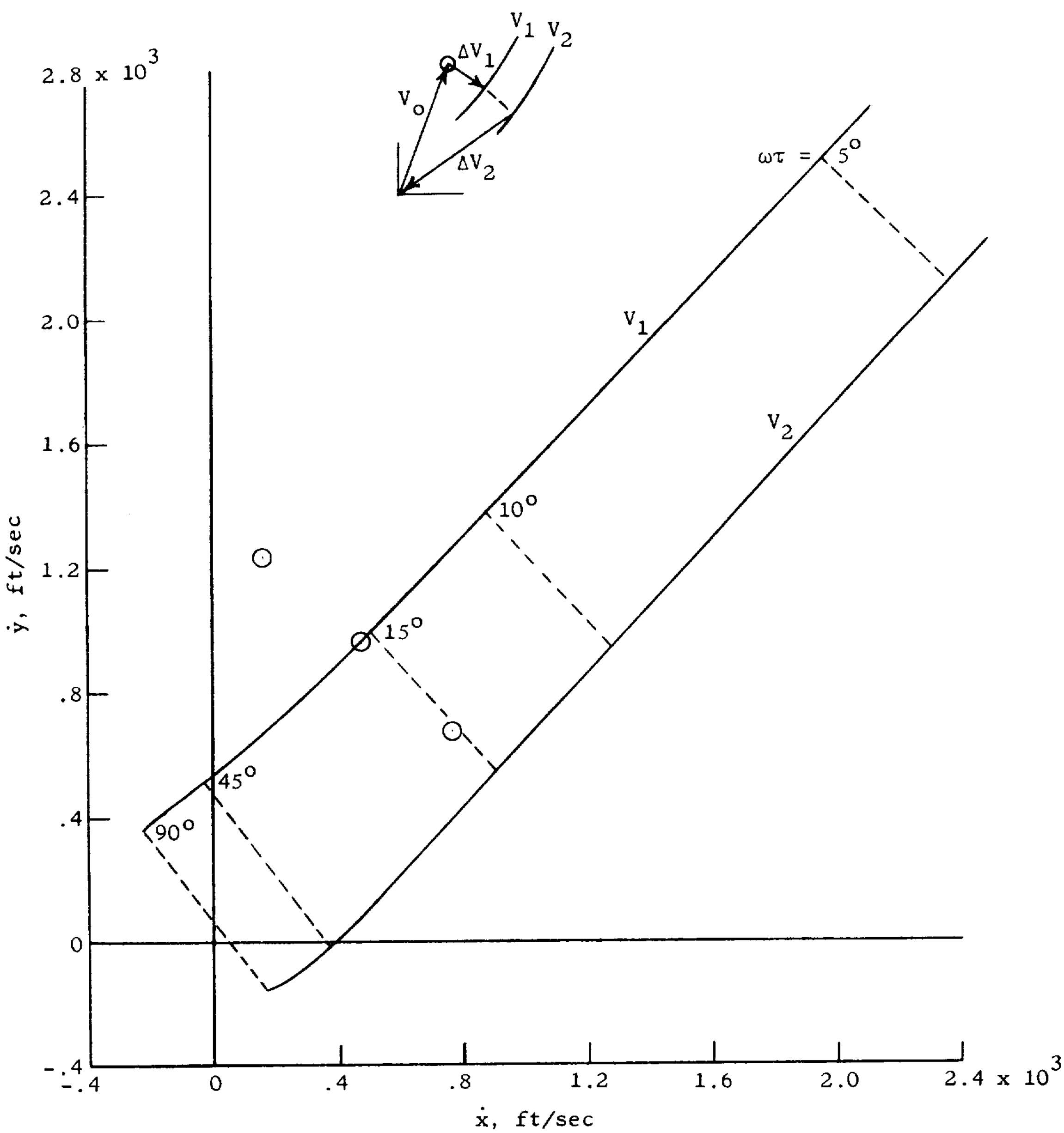


(c) Offcourse; $R_o\dot{\theta}_o$ decreased by 411.2 ft/sec.

Figure 5.- Concluded.



(a) Case I in rotating rectangular coordinates.

Figure 6.- Hodograph presentation of the velocity components for intercept and rendezvous.

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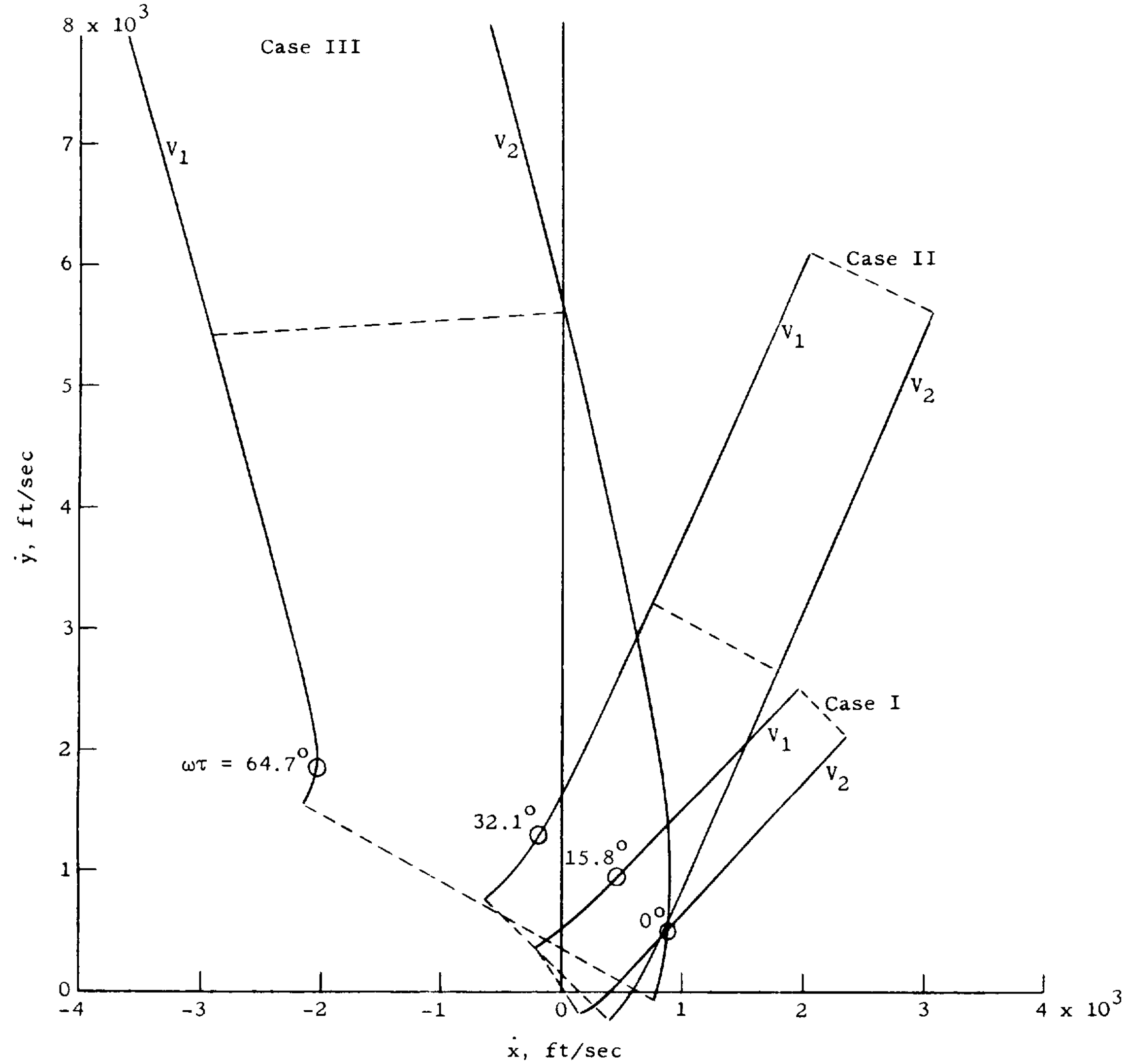
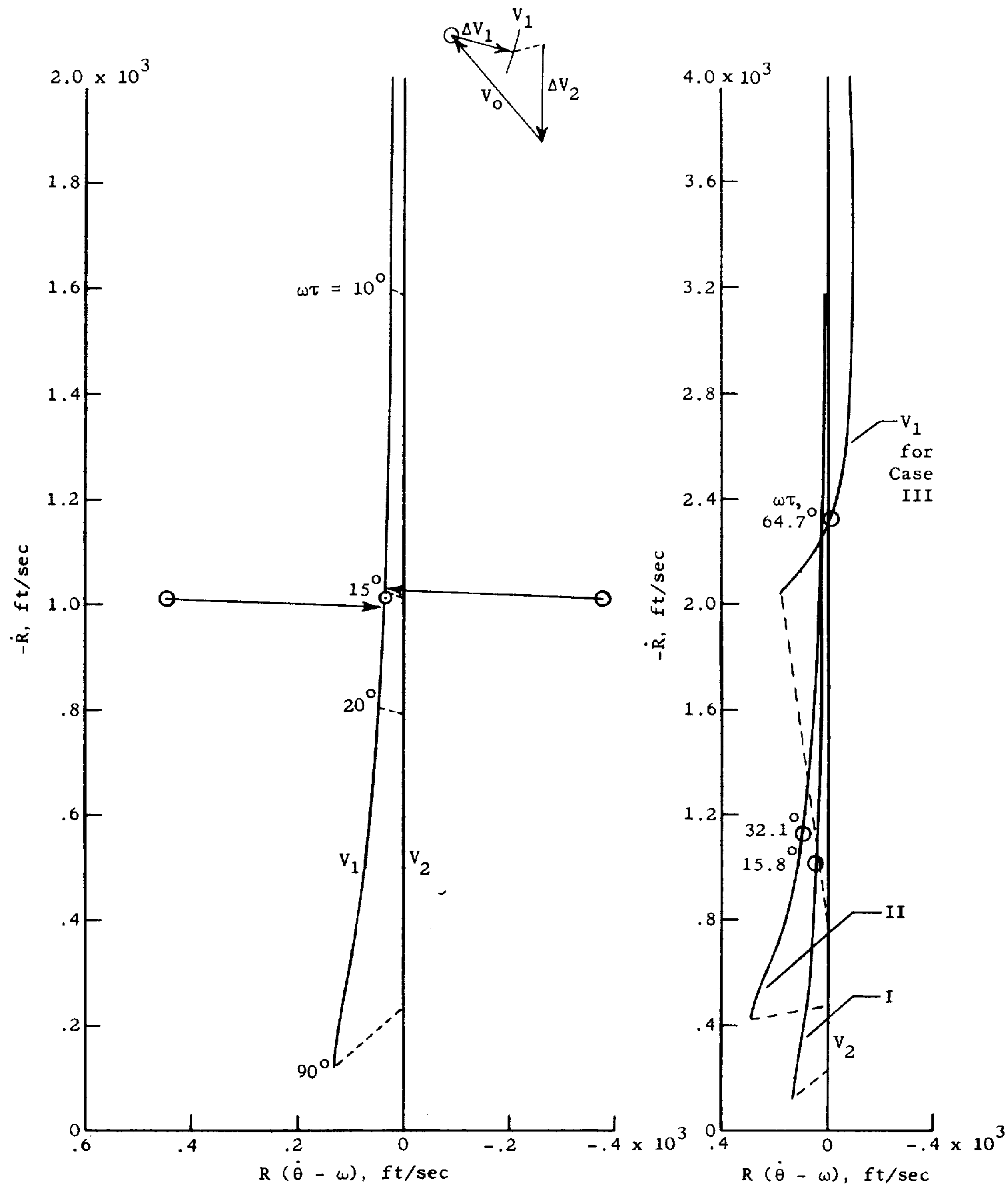


(b) Cases I, II, III in rotating rectangular coordinates.

Figure 6.- Continued.



(c) Case I in nonrotating polar coordinates.

(d) Cases I, II, III in nonrotating polar coordinates.

Figure 6.- Concluded

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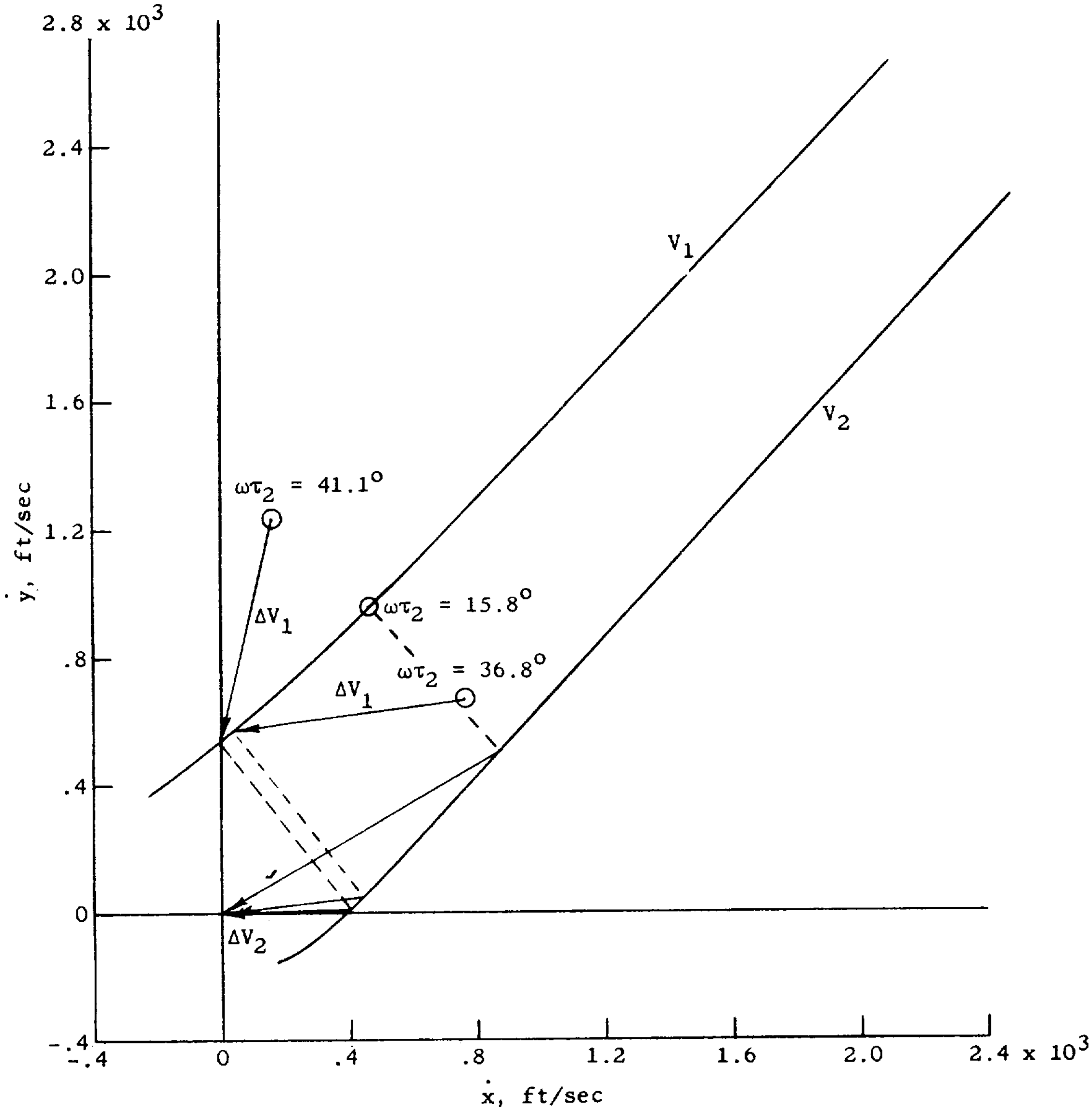

Figure 7.- Hodograph presentation of velocity corrections ΔV_1 and ΔV_2.